CLASS WILL START AT 5PM
Section 6a/b:
Markov Chain Dynamics and Control
Markov Chains

- Recall state-space, $\mathcal{X}$, can be simply a subset of $\mathbb{N}$ or $\{0\} \cup \mathbb{N}$ here.
- We suppose the transition probabilities are time-independent and Markov, with
  \[ P_{i,j} = P(\xi_{t+1} = j \mid \xi_t = i) = P([\xi_{t+1}]^{-1}(\{j\}) \mid [\xi_t]^{-1}(\{i\})). \]
- Let $[p_t]_j = P(\xi_t = j)$, and note that if $\#\mathcal{X} = n$, then $p_t \in S^n$ (that simplex) for all $t$.
- Dynamics of $\vec{p}_t = p_t$ process
  \[ p_{t+1} = P^T p_t. \]
- Note that $p_{t+k} = [P^T]^k p_t = [P^k]^T p_t$. 
Markov Chain Dynamics Classification Recollections

- **j is accessible from i** (i.e., \( i \rightarrow j \)) if there exists \( n \in [0, \infty) \) s.t. \( [P^n]_{i,j} > 0 \).
- **i communicates with j** (i.e., \( i \leftrightarrow j \)) if \( i \rightarrow j \) and \( j \rightarrow i \).
- We found that \( \leftrightarrow \) is an equivalence relation on \( \mathcal{X} \).
- Also, that implies that \( \leftrightarrow \) partitions \( \mathcal{X} \) into equivalence classes (disjoint subsets whose union is \( \mathcal{X} \)).
- A Markov chain is **irreducible** if \( i \leftrightarrow j \) \( \forall i, j \in \mathcal{X} \).
- **i \in \mathcal{X}** is absorbing if \( P_{i,j} = 0 \) \( \forall j \in \mathcal{X} \setminus \{i\} \).
The period of state \( i \in \mathcal{X} \), \( d(i) \), is the greatest common divisor of \( \{ n \in \mathbb{N} \mid [P^n]_{i,i} > 0 \} \).

(Special case: \([P^n]_{i,i} = 0\) for all \( n \) implies \( d(i) = 0 \).)

- \( d(i) = 1 \) for all \( i \in \mathcal{X} \) implies that the Markov chain is aperiodic.

- \( i \leftrightarrow j \) implies \( d(j) = d(i) \). That is, \( d(i) \) is the same for all the nodes in a communicating equivalence class.
Markov Chain Dynamics Classification

- The probability that the first occurrence of $j$ after $i$ occurs $n$ steps later is
  \[ f_{i,j}^n = P(\xi_{t+n} = j | \xi_t = i, \xi_{t+k} \neq j \ \forall k \in ]1, n-1[). \]

- $i$ is recurrent if $\sum_{i=1}^{\infty} f_{i,i}^n = 1$; otherwise, $i$ is transient.

**Theorem (1)**

*State $i$ is recurrent iff $\sum_{n=1}^{\infty} [P^n]_{i,i} = \infty$.***

**Theorem (2)**

*If $i \leftrightarrow j$, then $i$ is recurrent iff $j$ is recurrent (equivalently, $i$ is transient iff $j$ is transient).***

- Do proof?
Theorem (3)

If \((X, P)\) is finite (i.e., \(\#X\) is finite), and the Markov chain is irreducible, then all states are recurrent.

- Let \(n = \#X\). \(\Pi \in S^n\) is a stationary distribution of the MC if \(\Pi = P^T \Pi\).
- Note that a stationary distribution is an eigenvector of \(P^T\) corresponding to eigenvalue \(\lambda = 1\).
- A limit distribution is a stationary distribution. Indicate this.
Lemma (4)

Suppose $(\mathcal{X}, \mathcal{P})$ is finite, irreducible and aperiodic. Then, there exists $n_0 < \infty$ s.t. $[\mathcal{P}^n]_{i,j} > 0$ for all $i,j \in \mathcal{X}$ and all $n \geq n_0$.

Theorem (5)

Suppose $(\mathcal{X}, \mathcal{P})$ is finite, irreducible and aperiodic. Then, there exists a stationary distribution, $\Pi$, $A \in [0, \infty)$ and $\rho \in [0, 1)$ s.t.

$$|[\mathcal{P}^n]_{i,j} - \Pi_j| \leq A \rho^n \quad \forall i,j \in \mathcal{X}.$$ 

Let $(\mathcal{X}, \mathcal{P})$ be finite, irreducible and aperiodic. Fix initial distribution, $p_0$.

$$\lim_{t \to \infty} [p_t]_j = \lim_{t \to \infty} [(\mathcal{P}^t)^T p_0]_j = \text{Finish}.$$
Corollary (6)

Suppose $(\mathcal{X}, P)$ is finite, irreducible and aperiodic. Then, for any $p_0, p_t \to \Pi$, where $\Pi = P^T \Pi$, and $\Pi$ is the unique eigenvector of $P^T$ corresponding to eigenvalue $\lambda = 1$.

- Maybe do a proof. Provide counterexamples.
Consider a feedback policy, $\mu : \mathcal{X} \rightarrow U$, where $U$ is the set of possible controls (assumed finite and independent of $i \in \mathcal{X}$ for now).

For each $i \in \mathcal{X}$, let

$$P_{i,j}^{\mu} = P_{i,j}(\mu(i)) \quad \forall j \in \mathcal{X}.$$

Use UAV example to illustrate discounted-cost problem formulation.
With a bit of work, we’d obtain the DPE:

\[
V(i) = \min_{v \in U} \left\{ L(i, v) + \alpha \mathbb{E}_{\xi_0 = i} [V(\xi_1)] \right\} \\
= \min_{v \in U} \left\{ L(i, v) + \alpha \sum_{j \in X} P_{i,j}(v) V(j) \right\}.
\]

In the above example, this becomes the following equations.