MAE 288A
Assignment 5/Take-Home Final
Due Wednesday, 9 Dec., 2009

Note: You may find it helpful to write code in order to solve some of these problems. You may work in Matlab, C, C++, Java, Fortran, Maple, or Mathematica. Please include copies of any codes used.

1. (10) Consider the optimal control problem of exit type given by dynamics

\[ \dot{\xi}_t = u_t, \quad \xi_0 = x \in (-1, 1), \]

payoff

\[ J(x; u.) = \int_0^\tau \rho + (u_t^2/2) \, dt, \]

and value

\[ V(x) = \inf_{u \in U} J(x; u.), \]

where \( \rho = 1 \), \( \tau = \inf\{t \geq 0 | \xi_t \notin (-1, 1)\} \), and \( U \) is the set of \( L_2 \) controls taking values in \( U = \mathbb{R} \), as indicated in class. Construct the grid-based numerical scheme as discussed in class to numerically obtain the value function. You will need to be a bit careful about the time and space step-sizes in order to ensure the you obtain a reasonable solution approximation. Use at least 21 grid points over \([-1, 1]\). You do not need to take limits as time and space step sizes converge to zero. (Reference: Bardi/Capuzzo-Dolcetta, Appendix A.)

2. (5) Define the operator, \( S_\delta : C_B(\mathbb{R}^n) \rightarrow C_B(\mathbb{R}^n) \) (where \( C_B \) denotes the space of bounded, continuous functions)

\[ S_\delta[\phi](x) = \sup_{u \in L_2((0,\delta);\mathbb{R}^m)} \left\{ \int_0^\delta L(\xi_t, u_t) \, dt + \phi(\xi_\delta) \right\}, \]

where \( \xi \) satisfies

\[ \dot{\xi}_t = f(\xi_t, u_t), \quad \xi_0 = x \in \mathbb{R}^n, \]
and we make sufficient assumptions on $f$ and $L$ to guarantee that the operator is well-defined. Suppose $\phi, \psi \in C_B(\mathbb{R}^m)$. Prove that

$$S_\delta[\phi \oplus \psi] = S_\delta[\phi] \oplus S_\delta[\psi],$$

where $\oplus, \otimes$ denote max-plus addition and multiplication.

3. (10) Consider the control problem

$$\dot{\xi}_t = f(\xi_t, u_t), \quad \xi_s = x \in \mathbb{R}^n$$

$$J(s, x, u.) = \int_s^T L(\xi_r, u_r) \, dr + \psi(\xi_T)$$

$$V(s, x) = \inf_{u \in U_s} (s, x, u.),$$

where $U_s$ is the set of $L_2$ controls taking values in compact set $U \subset \mathbb{R}^m$. You may assume that $f$ and $L$ are sufficiently nice so that solutions of the dynamics exist and are continuous for all $u$, and such that $V$ is everywhere finite and continuous. The associated HJB PDE is

$$0 = W_s(s, x) + \min_{v \in U}[\nabla_x W(s, x) \cdot f(x, v) + L(x, v)] \quad (s, x) \in (0, T) \times \mathbb{R}^n.$$ 

Prove that $V$ is a viscosity subsolution of its associated HJB PDE. The proof will be somewhat analogous to the proof in class that $V$ is a viscosity supersolution. Instead of choosing the constant control that we used in that proof, you will use a control that is $\varepsilon$-optimal for the dynamic programming principle (DPP). (Make whatever continuity assumptions you deem appropriate.)

4. (5) Let $B$ be given by

$$B = \begin{bmatrix} 0 & 1 & -1 & -2 \\ -1.5 & -1 & 0.6 & 1.5 \\ -1.7 & -1 & -0.2 & 1 \\ -2.7 & -2 & -1.2 & -0.2 \end{bmatrix}.$$ 

Compute the max-plus eigenvector of $B$ corresponding to max-plus eigenvalue 0. (That is, find the solution to $a = B \otimes a$.) Use the max-plus power method.