MAE 288A
Assignment 3
Due 9pm, 21 May

Problems to hand in. (Not all problems may be graded.)
All work and explanations must be included for full credit.
In the case of problems where code was used, include the code.
We can only grade matlab and python.
More problems will NOT be added to the assignment.

1. (10) Let the space $C_{B,2}(\mathbb{R}^n)$ be given by

$$C_{B,2}(\mathbb{R}^n) \doteq \{y : \mathbb{R}^n \to \mathbb{R} \mid \exists M_y < \infty \text{ such that } |y(x)| \leq M_y(1+|x|^2) \ \forall x \in \mathbb{R}^n\}.$$

Is this a vector space? Define

$$\|y\| = \sup_{x \in \mathbb{R}^n} \frac{|y(x)|}{1+|x|^2}.$$

Show that $\|\cdot\|$ maps into $[0,\infty)$. Is it a norm? (Support your answer.)

2. (4) Consider the iteration

$$x_{n+1} = \alpha x_n - 0.5(\arctan(x_n) + 1).$$

For what values of α, do the conditions of the Banach Fixed-Point Theorem hold? When they hold, what equation will the iteration limit solve?

3. (10) Consider the discounted-cost, infinite time-horizon control problem with dynamics

$$\xi_{t+1} = [1 - \bar{u}(\xi_t)]\, \xi_t + w_t$$

where ξ_t, w_t are scalar-valued, and $\bar{u} : \mathbb{R} \to \mathbb{R}$. Suppose $U = \{1/2, 3/4\}$. Let the w_t be IID normal random variables with mean 0 and variance 1. Let the cost criterion be

$$J(x, \bar{u}) = E_{\xi_0 = x}\Big\{ \sum_{t=0}^{\infty} \alpha^t \Big[\tfrac{1}{2}\xi_t^2 + \tfrac{1}{2}(\bar{u}(\xi_t))^2 \Big] \Big\}$$

where $\alpha = 3/4$. Write down the DPE for this problem. Starting with initial guess, $V_0 \equiv 0$, perform two steps in a value iteration/successive approximations/ Banach Fixed Point Theorem approach to solving the DPE. That is, compute V_1 and V_2. If you stopped after only these two steps, what would be your estimate of the optimal feedback control?

4. (10) Consider the discounted-cost, infinite time horizon problem with dynamics

$$\begin{aligned} \xi_{t+1} &= A\xi_t + B\bar{u}(\xi_t) + w_t, \\ \xi_0 &= x \end{aligned}$$

and cost criterion

$$J(x; u.) = \mathbf{E}\left\{ \sum_0^\infty \alpha^t \left[\frac{1}{2}\xi_t^T C \xi_t + \frac{1}{2}\bar{u}^T(\xi_t) D \bar{u}(\xi_t) \right] \right\}.$$

Suppose the noise process consists of independent, identically distributed, normal random variables with mean 0 and covariance Q. Consider the special case

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = 2,$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \alpha = 0.5$$

Starting with $V_0 \equiv 0$, compute three steps in a value iteration, indicating the approximate control you would get at each step. (More specifically, you only compute three approximate controls, with the first being trivial.)

5. (6) Give examples of discrete-time Markov chains (with state space no smaller than 3 and no larger than 5) which are:

   (a) Periodic with period 2.
   (b) Irreducible and aperiodic.
   (c) Decomposable into two equivalence classes of communicating states.