**MAE 288A**
**Assignment 2**
**Due 9pm, 7 May**

**Problems to hand in. (Not all problems may be graded.)**
**All work and explanations must be included for full credit.**
**In the case of problems where code was used, include the code.**
**We can only grade matlab and python.**
**More problems will likely be added to the assignment.**

1. (8) Suppose we have a graph with nodes $\mathcal{G} = \{1, 2, \dots n\}$. Suppose edges exist between every pair of nodes, i.e., the set of edges is $\mathcal{E} = \{(x, y) \,|\, x, y \in \mathcal{G}\}$. How many paths of length $K$ are there? That is, how many paths of the form $\{(x_0, x_1), (x_1, x_2), \dots (x_{K-1}, x_K)\}$ where each $x_k \in \mathcal{G}$ are there? (It's fine if a path passes through the same node multiple times, where that includes cases such that $x_k = x_{k+1}$.) Suppose we assign costs, $c(x, y)$ to every edge, and a terminal cost, $\phi(x)$ to every node. Consider the backward DP algorithm given by setting $V(K, x) = \phi(x)$ for all $x \in \mathcal{G}$, and then, for each $k = K-1, K-2, \dots 0$, computing

$$V(k, x) = \min_{y \in \mathcal{G}} \big\{ c(x, y) + V(k+1, y) \big\} \quad \forall x \in \mathcal{G}.$$

How many times is a minimum computed by this DP in order to finally obtain $V(0, \cdot)$? In contrast, indicate how many such operations would be required by an admittedly foolish brute force enumeration of all possible paths, followed by a minimization over each of their total costs? For concreteness, suppose that a minimum over $M$ real numbers is obtained by performing a minimum operation on two real numbers $M-1$ times.

2. (10) Complete the second half of the proof of the finite time-horizon dynamic programming principle theorem given in class. You may assume the existence of an optimal controller (rather than using $\epsilon$-optimal controls) to make it a bit easier if you like. Hint: In that case, one method is as follows. Consider an optimal control for the entire period, $]s, T[$.

Then break the resulting costs for that entire period into the sum of the costs over the set of times $]s, t-1[$ and the sum of the costs over the set of times $]t, T[$. The cost over the times $]t, T[$ with this controller and any $\xi_t$ is certainly higher than $V(t, \xi_t)$.

3. (10) Consider the finite time-horizon stochastic control problem with dynamics and initial condition

$$\xi_{t+1} = A\xi_t + Bu_t + b + w_t, \qquad \xi_s = x \in \mathbb{R}^n,$$

and payoff

$$J(t_0, x; u_\cdot) = \mathbf{E}\left\{ \sum_{t=s}^{T-1} \left[\tfrac{1}{2}\xi_t^T C \xi_t + \tfrac{1}{2} f u_t^T D u_t\right] + \tfrac{1}{2}\xi_T^T F \xi_T \right\},$$

and where the control, $u_\cdot$, seeks to minimize the payoff. Let the value function be denoted by $V(t, x)$ for all $t \in ]s, s+1, \ldots, T[$ and $x \in \mathbb{R}^n$. Let the control take values in $\mathbb{R}^n$, and let the noise process consist of independent, identically distributed, normal random variables (taking values in $\mathbb{R}^n$) with mean zero and covariance matrix $Q_w$. Suppose $V(t, x)$ takes the form $V(t, x) = \frac{1}{2}(x - \bar{x}_t)^T R_t (x - \bar{x}_t) + r_t$ for all $t, x$. Using the finite time-horizon dynamic programming equation, find backward recursions for $R_t$, $\bar{x}_t$, and $r_t$. Assume that $A, B, b, C, D, F$ are appropriately dimensioned vectors, matrices and/or scalars. You may assume $C, D, F$ are positive-definite, symmetric, and that where inverses are needed the relevant matrices are nonsingular. Note: Some people find it helpful to expand the $V$ form as

$$\begin{aligned} V(t, x) &= \tfrac{1}{2}(x - \bar{x}_t)^T R_t (x - \bar{x}_t) + r_t \\ &= \tfrac{1}{2} x^T R_t x - \tfrac{1}{2} x^T R_t \bar{x}_t - \tfrac{1}{2} \bar{x}_t^T R_t x + \left[\tfrac{1}{2}\bar{x}_t^T R_t \bar{x}_t + r_t\right] \end{aligned}$$

when attempting to match the quadratic, linear and zeroth-order coefficients.

4. (5) Consider the special case given by

$$A = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q_w = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \qquad D = 2, \qquad F = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let $T = 10$ and $s = 0$. Find $V(t, x)$ and the optimal feedback control for $t \in \,]s, T-1[\, \doteq \{0, 1, \ldots 10\}$.

5. (2) Let $\mathbb{R}^- \doteq \mathbb{R} \cup \{-\infty\}$. Let $\tilde{\mathcal{C}}(\mathbb{R}^n; \mathbb{R}^-)$ denote the space of convex functions on $\mathbb{R}^n$ taking values in $\mathbb{R}^-$. Let $f, g \in \tilde{\mathcal{C}}(\mathbb{R}^n; \mathbb{R}^-)$. Let $h(x) \doteq f(x) + g(x)$ for all $x \in \mathbb{R}^n$. Prove that $h \in \tilde{\mathcal{C}}$.