MAE 288A
Assignment 1
Due 9pm, 22 Apr.

Problems to hand in (Not all problems may be graded.)

All work and explanations must be included for full credit.

1. (5) let $\Omega = \{a, b, c, d, e\}$. Let $X(a) = 1$, $X(b) = 1$, $X(c) = 2$, $X(d) = 2$, $X(e) = 3$. Let $Y(a) = 4$, $Y(b) = 5$, $Y(c) = 6$, $Y(d) = 4$, $Y(e) = 5$. Suppose $P(\{a\}) = P(\{b\}) = P(\{c\}) = 0.25$, and $P(\{d\}) = P(\{e\}) = 0.125$. What are $P(X \in \{1, 2\})$, $P(Y \in \{5, 6\})$, $P(X \in \{1, 2\}, Y \in \{5, 6\})$ and $P(X \in \{1, 2\} \,|\, Y \in \{5, 6\})$? Also, are X and Y independent?

2. (5) X_1 and X_2 are scalar random variables with joint density function $f : \mathbb{R}^2 \to \mathbb{R}$ such that
$$P(X_1 \leq u_1, X_2 \leq u_2) = \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} f(y_1, y_2)\, dy_2\, dy_1$$
for all $(u_1, u_2) \in \mathbb{R}^2$. Prove that
$$P(X_1 \in (l_1, u_1], X_2 \in (l_2, u_2]) = \int_{l_1}^{u_1} \int_{l_2}^{u_2} f(y_1, y_2)\, dy_2\, dy_1$$
for all $-\infty < l_1 \leq u_1 < \infty$ and $-\infty < l_2 \leq u_2 < \infty$. (You may use the result obtained in class if it is helpful.)

3. (5) Suppose X, Y and Z are independent random variables with ranges in $\mathbb{R}^k$, $\mathbb{R}^m$ and $\mathbb{R}^k$, resectively. Suppose $G \in C(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^m)$, and that $U = G(X, Y)$. Prove that U and Z are independent.

4. (5) Let $X = (X_1, X_2)^T$ be a random variable taking values in $\mathbb{R}^2$, with density function given by
$$f(x_1, x_2) = \begin{cases} x_1 x_2 \exp\left[\frac{-1}{2}(x_1^2 + x_2^2)\right] & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$
What is $P(X_1 \geq 1 \,|\, X_2 \geq 1)$? Are X_1 and X_2 independent?

5. (10) Consider the Gauss-Markov process given by $\xi_{k+1} = A\xi_k + Bw_k$ for all $k \in \mathcal{K}$ where $\mathcal{K} \doteq \{0, 1, 2 \dots\}$ where $\{w_k\}_{k\in\mathcal{K}}$ is an IID sequence of random variables with $w_k \sim \mathcal{N}(0, \sigma_w^2)$ for all $k \in \mathcal{K}$. Let $\xi_0 \sim \mathcal{N}(\bar{x}_0, C_0)$, and suppose ξ_0 is independent of w_k for all k. Suppose

$$A = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

If possible, find a value of C_0 such that $C_k = C_0$ for all k. If that is not possible, indicate why. How about $C_{2k} = C_0$ for all k?