1. (5) Consider the discrete-time stochastic control problem with dynamics
\[ \xi_{t+1} = A\xi_t + Bu_t + w_t, \quad \xi_{t_0} = x \]
and cost criterion
\[ J(t_0, x; u) = E \left\{ \sum_{t=t_0}^{T-1} \left[ \frac{1}{2} \xi_t^T C\xi_t + \frac{1}{2} u_t^T D u_t \right] + \frac{1}{2} \xi_T^T F\xi_T \right\}. \]
Let \( u \) be a minimizing control, and let the value function be denoted by \( V(t, x) \) for all \( t \in \{t_0, t_0 + 1, \ldots, T\} \) and \( x \in \mathbb{R}^n \). Let the control take values in \( \mathbb{R}^k \), and let the noise process consist of independent, identically distributed, normal random variables (taking values in \( \mathbb{R}^n \)) with mean 0 and covariance matrix \( Q \). Consider the special case
\[ A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 2 \]
\[ F = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}. \]
Let \( T = 4 \) and \( t_0 = 1 \). Find \( V(t, x) \) and the optimal feedback control for \( t \in \{1, 2, 3\} \).

2. (10) Consider the discrete-time stochastic control problem with dynamics
\[ \xi_{t+1} = A\xi_t + Bu_t + b + w_t, \quad \xi_{t_0} = x, \]
and cost criterion
\[ J(t_0, x; u) = \mathbb{E} \left\{ \sum_{t=t_0}^{T-1} \left[ \frac{1}{2} \xi_t^T C \xi_t + \frac{1}{2} u_t^T D u_t \right] + \frac{1}{2} \xi_T^T F \xi_T \right\}. \]

Let \( u \) be a minimizing control, and let the value function be denoted by \( V(t, x) \) for all \( t \in \{t_0, t_0 + 1, \ldots, T\} \) and \( x \in \mathbb{R}^n \). Let the control take values in \( \mathbb{R}^n \), and let the noise process consist of independent, identically distributed, normal random variables (taking values in \( \mathbb{R}^n \)) with mean zero and covariance matrix \( Q \). Suppose \( V(t, x) \) takes the form
\[ V(t, x) = \frac{1}{2} x^T P_t x + \lambda_t^T x + r_t \]
for all \( t, x \). Find backward recursions for \( P_t, \lambda_t, \) and \( r_t \). Assume that \( A, B, C, D, F \) are appropriately dimensioned matrices and the \( b \in \mathbb{R}^n \) is fixed and known. You may assume \( C, D, F \) are positive-definite, symmetric, and that where inverses are needed the relevant matrices are nonsingular.

3. (5) Let \( f_1, f_2 \) be convex functions mapping \( \mathbb{R}^n \) into \( \mathbb{R} \). Let \( g = f_1 \lor f_2 \). That is, let \( g(x) = f_1(x) \lor f_2(x) = \max\{f_1(x), f_2(x)\} \) for all \( x \in \mathbb{R}^n \). Prove that \( g \) is convex.

4. (5) Suppose \( f: \mathbb{R} \rightarrow \mathbb{R} \) is convex. Let \( \bar{x} \in \text{argmin}\{f\} \). Is \( f \) monotonically increasing on \( [\bar{x}, \infty) \)? If so, prove it. If not, provide a counterexample.

5. (10) Consider an inventory-control problem formulated as in class. The dynamics are \( \xi_{t+1} = \xi_t + u_t - w_t \) with initial condition \( \xi_s = x \in [0, \infty) \). The \( \{w_t\}_{t=s}^{T-1} \) noise process consists of IID random variables, each with uniform distribution over \([0, 6]\). The control, \( u_t \) may take values in \([0, \infty) \). The payoff, given feedback control policy \( \{\mu_t\}_{t=s}^{T-1} \in \mathcal{M}_{s,T-1}^F \) is given by
\[ J(s, x, \mu) = \mathbb{E}_{\xi_s = x} \left\{ \sum_{t=s}^{T-1} L(\xi_t, \mu_t(\xi_t)) + \phi(\xi_T) \right\}, \]
\[ L(x, v) = (p_i x) \lor (-p_b x) + cv, \quad \phi(x) = (d^+ x) \lor (-d^- x), \]
where for ease of computations, we take \( p_i = p_b = 5, \ d^+ = d^- = 8 \) and \( c = 3 \). Perform backward dynamic programming to obtain the value function at time \( T - 1 \), i.e., \( V(T - 1, \cdot) \). The complexity of hand computations prohibits further back propagation here. While performing these computations, specifically indicate \( \mathbb{E}[V(T, y - w_{T-1})] \).
as a function of $y$. Also indicate $\arg\min_{y \in \mathbb{R}} \{cy + \mathbb{E}[V(T, y - w_{T-1})]\}$. What is the optimal control policy $\mu_{T-1}(\cdot)$ (as a function of $x$)?