MAE 288A
Assignment 1
Due Tuesday, 24 April

Problems to hand in (Not all problems may be graded.)

All work and explanations must be included for full credit.

1. (5) let $\Omega = \{a, b, c, d, e\}$. Let $X(a) = 1$, $X(b) = 1$, $X(c) = 2$, $X(d) = 2$, $X(e) = 3$. Let $Y(a) = 4$, $Y(b) = 5$, $Y(c) = 6$, $Y(d) = 4$, $Y(e) = 5$. Suppose $P(\{a\}) = P(\{b\}) = P(\{c\}) = 0.25$, and $P(\{d\}) = P(\{e\}) = 0.125$. What are $P(X \in \{1, 2\})$, $P(Y \in \{5, 6\})$, $P(X \in \{1, 2\}, Y \in \{5, 6\})$ and $P(X \in \{1, 2\} | Y \in \{5, 6\})$, Also, are $X$ and $Y$ independent?

2. (5) $X_1$ and $X_2$ are scalar random variables. Suppose there exists a continuous function, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$P(X_1 \leq u_1, X_2 \leq u_2) = \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} f(y_1, y_2) \, dy_2 \, dy_1$$

for all $(u_1, u_2) \in \mathbb{R}^2$. Prove that

$$P(X_1 \in [l_1, u_1], X_2 \in [l_2, u_2]) = \int_{l_1}^{u_1} \int_{l_2}^{u_2} f(y_1, y_2) \, dy_2 \, dy_1$$

for all $-\infty < l_1 \leq u_1 < \infty$ and $-\infty < l_2 \leq u_2 < \infty$. (You may use the result obtained in class if it is helpful.)

3. (10) Suppose $\Sigma_i$ are $\sigma$-algebras on sample space $\Omega$ for $i \in \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers. Prove that $\Sigma = \bigcap_{i=1}^{\infty} \Sigma_i$ is a $\sigma$-algebra on $\Omega$.

4. (5) Suppose $w$ is a scalar normal random variable with mean zero and variance, $q$. Consider $E[\exp\{kw^2/2\}]$ where $k$ is a given positive constant. For what values of $k$ is this finite? When it is finite, what is it?

5. (5) Consider the Gauss-Markov process $\xi_{k+1} = \alpha \xi_k + w_k$ for all $k \in K$ where $K = \{0, 1, 2, \ldots\}$ and $\{w_k\}_{k \in K}$ is an IID sequence of random
variables with \( w_k \sim \mathcal{N}(0, 2) \) for all \( k \in K \). If possible, obtain a value of \( \alpha \) such that the (long-term) steady-state distribution of \( \xi_k \) is normal with covariance \( Q = 4? \) If it is not possible, indicate why. How about \( Q = 1? \)

6. (5) Suppose we have a scalar-valued model for a dynamical system with noise given as \( \dot{\xi}_t = f(\xi_t, t) + \sigma B_t \) with initial state \( \xi_0 = x_0 \in \mathbb{R} \), where \( \sigma = 3 \), \( f(x, t) = -2x + e^{-t} \) for all \( x \in \mathbb{R} \) and \( B_t \) is a Brownian motion process. (The correct way to write this is as \( d\xi_t = f(\xi_t, t) dt + \sigma dB_t \), but the above form is fine for our purposes here.) Let the nominal trajectory (i.e., the trajectory in the absence of noise) be given by \( \dot{\tilde{\xi}}_t = f(\tilde{\xi}_t, t) \), with \( \tilde{\xi}_0 = x_0 \) Letting \( \tilde{\xi}_k = \xi_{k\delta} - \tilde{\xi}_{k\delta} \) for \( k \in \mathbb{N} \), with \( \delta << 1 \), Obtain an approximate model for \( \tilde{\xi}_k \) in the form \( \tilde{\xi}_{k+1} = A_k \tilde{\xi}_k + B_k w_k \), and provide expressions for \( A_k \) and \( B_k \). Note that for a scalar-valued Brownian motion, \( B_s - B_t \sim \mathcal{N}(0, t - s) \) for \( s < t \). You do not need to provide an error analysis indicating the mismodeling errors induced by this model.

7. (5) Suppose we have a graph with nodes \( G = \{1, 2, \ldots, n\} \). Suppose edges exist between every pair of nodes, i.e., \( E = \{(x, y) | x, y \in G\} \). How many paths of length \( K \) are there? That is, how many paths of the form \( \{(x_0, x_1), (x_1, x_2), \ldots (x_{K-1}, x_K)\} \) where each \( x_k \in G \) are there? (It’s fine if a path passes through the same node multiple times, where that includes cases such that \( x_k = x_{k+1} \).) Suppose we assign costs, \( c(x, y) \) to every edge, and a terminal cost, \( \phi(x) \) for every node. Consider the backward DP algorithm given by setting \( V(K, x) = \phi(x) \) for all \( x \in G \), and then, for each \( k = K - 1, K - 2, \ldots, 0 \), computing

\[
V(k, x) = \min_{y \in G} \left\{ c(x, y) + V(k + 1, y) \right\} \quad \forall x \in G.
\]

How many times is a min between two numbers computed by this DP in order to finally obtain \( V(0, \cdot) \)? (For our purposes here, you may assume that a min over \( n \) numbers is obtained as a sequence of min operations between two numbers in the most efficient manner. Also, if you like, you are free to assume \( n = 2^m \) for some natural number \( m \).) How many such operations would be required by an admittedly foolish brute force enumeration of all possible paths, and a minimization over their costs?