MAE 285B  
Take-Home Final/Assignment 4  
Due Thursday, 3/22/07  

Note: You may find it helpful to write code in order to solve some of these problems. You may work in Matlab, C, C++, Java, Fortran, Maple, or Mathematica. Please include copies of any codes used.

1. (5) Suppose $X$ and $U$ are finite, and that $f, g : X \times U \to \mathbb{R}$. Show that

$$\left| \min_{u \in U} f(i, u) - \min_{v \in U} g(i, v) \right| \leq \max_{u \in U} |f(i, u) - g(i, u)|$$

for all $i \in X$.

2. (10) Consider the optimal control problem with dynamics

$$\dot{\xi} = A\xi_t + Bu_t,$$
$$\xi_s = x \in \mathbb{R}^n$$

and payoff

$$J(s, x; u.) = \int_s^1 \left[ \frac{1}{2} \xi_t^T C\xi_t + \frac{1}{2} u_t^T D u_t \right] + \frac{1}{2} (\xi_1 - \hat{x})^T Q(\xi_1 - \hat{x}).$$

(Note that $\hat{x}$ is a fixed vector – not time-dependent.) Let $u.$ be a minimizing control, and let the value function be denoted by $V(s, x)$ for all $s \in [0, 1]$. Let the control take values in $\mathbb{R}^m$. You may assume all matrices are as nice as needed. In particular, let $C$, $D$, and $Q$ be positive definite, symmetric matrices. Suppose $V(s, x)$ takes the form $V(s, x) = \frac{1}{2} (x - \bar{x}_s)^T R_s (x - \bar{x}_s) + r_s$ for all $s, x$. Find differential equations and terminal conditions for $R_s$, $\bar{x}_s$ and $r_s$.

3. (5) Consider the special case

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 1,$$
$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find $V(0, x)$. 

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4. (10) Consider the PDE problem

\[ 0 = -\left[\frac{1}{2} - \frac{1}{2}||\nabla V||^2\right] \quad \forall x \in \Omega = (-2, 2) \times (-1, 1) \]

\[ V(x) = 0 \quad \forall x \in \partial \Omega. \]

Compute (the state-space components) of the characteristics emerging from the boundary (excluding the corners) and the corresponding value function along those characteristics. Discuss what happens to the projection of these characteristics onto the state space. What kind of trouble might you have in determining \( V \) from these characteristics?

5. (5) What control problem does the HJB PDE problem of the previous problem correspond to. Define all terms clearly.

6. (5) Let \( \mathbb{R}^- = \mathbb{R} \cup \{-\infty\} \). We say \( \phi : \mathbb{R}^n \to \mathbb{R}^- \) is uniformly semi-convex with constant \( c \) if \( \phi(x) + (c/2)||x||^2 \) is convex. Let \( \mathcal{S}_c(\mathbb{R}^n) \) be the space of uniformly semiconvex functions on \( \mathbb{R}^n \) with constant \( c \). We say a space, \( \mathcal{X} \), is a max-plus vector space (i.e., a max-plus module) if \( u, v \in \mathcal{X} \) and \( a, b \in \mathbb{R}^- \) imply that \( a \otimes u \oplus b \otimes v \in \mathcal{X} \). Here \( a \oplus b = \max\{a, b\} \) and \( a \otimes b = a + b \). Is \( \mathcal{S}_c(\mathbb{R}^n) \) a max-plus vector space? Support your answer.