Note: You may find it helpful to write code in order to solve some of these problems. You may work in Matlab, C, C++, Java, Fortran, Maple, or Mathematica. Please include copies of any codes used.

1. (5) Give examples of Markov chains (with state space no smaller than 3 and no larger than 4) which are:

   (a) Irreducible and aperiodic.

   (b) Decomposable into two equivalence classes of communicating states.

2. (10) Consider the gambling problem discussed in class with the probability of winning being $p = 0.4$ (and the probability of losing being $q = 1 - p = 0.6$). Suppose the gambler bets $1 at each time, that the gambler starts with $g_0 = $2, and that the opponent starts with $o_0 = $4. If either person goes bankrupt, they stay bankrupt for all time; otherwise the bet remains at $1 at each time. Estimate the probabilities of the player eventually going bankrupt and of the opponent eventually going bankrupt. What is the expected payoff to the gambler? Obtain the same information in the case where the gambler bets the maximum allowable (such that neither player can ever have a negative balance) at each step. Again estimate the probabilities of each player going bankrupt and the expected payoff to the gambler. Should the gambler bet $1 each time, or the maximum allowable amount?

Now, vary $o_0$ from $1 to $10, and compare the expected payoffs to the gambler for each of the two control policy options.

3. (10) Consider a Markov chain control problem where $X = \{1, 2, 3\}$ and $U = \{a, b, c\}$. Let the transition probability matrices be

$$P(a) = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 3/4 & 1/4 \end{bmatrix}, \quad P(b) = \begin{bmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{bmatrix}$$
Let the running cost be

\[ l(i, u) = \begin{cases} 
-2i & \text{if } u = a \\
1 - 2i & \text{if } u = b, c.
\end{cases} \]

Let the discount factor be \( \alpha = 0.8 \). Use policy iteration starting with \( \bar{u}(i) = c \) for all \( i \) to solve the problem.

4. (5) Suppose \( X \) and \( U \) are finite, and that \( f, g : X \times U \rightarrow \mathbb{R} \). Show that

\[ |\min_{u \in U} f(i, u) - \min_{v \in U} g(i, v)| \leq \max_{u \in U} |f(i, u) - g(i, u)| \]

for all \( i \in X \).