1. (10 points) Suppose our spacecraft is in a polar orbit (one where the inclination is $i = \pi/2$) above some planet. The orbit is circular with semi-major axis $a = 25000$ km. You may assume that the data is given in a PCI coordinate system, and you may set $\Omega = \omega = 0$. We would like to move the spacecraft to a mapping orbit. This will also be polar and circular. However, in this case, we would like the orbit period to be such that the point at which the ground-track crosses the equator will precess by approximately 120 km each orbit. (That is, if the spacecraft crosses the equator at $\lambda$ longitude on one pass, then it should pass over a point roughly 120 km east or west of that point on its next orbit.) Determine some maneuvers which will achieve that goal, while minimizing the total fuel expenditure. (You do not need to prove that they minimize the fuel expenditure!) Specifically, indicate the location of the burns in terms of true anomaly, $\nu$, and in terms of PCI position. Also provide each $\Delta \vec{v}$ in both the orbit-plane/perifocal and the PCI coordinates. Lastly, what is the total $\Delta \vec{v}$ required? The relevant data is as follows.

- Planet $Gm$: $\mu_p \simeq 600000.0 \text{ km}^3/\text{sec}^2$.
- Planetary radius: $r_p \simeq 6500$ km.
- Period of planet rotation: $T_p \simeq 20000$ sec.

2. (17 points) Consider a satellite in an orbit around the earth given by

- $a = 26561.82$ km
- $e = 0.6$
- $i = \pi/4$ radians
- $\Omega = -\pi/2$ radians
- $\omega = \pi/2$ radians
- $\tau = 0$ sec.

Plot each of the orbit-plane/perifocal components of the orbit as a function of time, over the time period from $t = 0$ to $t = 86164.1$ seconds, preferably all in one plot window (although three separate plots are acceptable). Make sure that the curves are clearly labeled. Plot the path of the satellite in orbit-plane/perifocal coordinates over that time period as a curve in three dimensions. (Obviously, the third component should be zero in these coordinates!) Repeat this procedure for the case of the ECI positions. Lastly, repeat the procedure for the case of the ECEF positions, where we suppose the ECI and ECEF coordinates are aligned at $t = 0$. Use the earth $\mu_e = Gm_e \simeq 398603. \text{ km}^3/\text{sec}^2$. 
3. (8 points) Consider the same satellite and time period as in the previous problem. Plot the geocentric latitude versus the longitude of the point on the earth (spherical model of the earth) that is directly below the satellite at each moment. That is, plot the satellite groundtrack (with the spherical model). Suppose there is a groundstation located at 242.8 degrees longitude, 32.7 degrees geodetic latitude. What is the ECEF position of the groundstation? Plot each of the ENU components of the satellite position as functions of time, preferably all in one plot window (although three separate plots are acceptable). Make sure that the curves are clearly labeled. The elevation in the ENU system is the angle above the local horizontal. The azimuth is the angle between the projection of the ENU position onto the local horizontal plane and local north (the north basis vector direction). Plot the elevation and azimuth of the satellite relative to the groundstation over the time period such that the elevation is positive.

4. (10) Although the state dimension and dynamics complexity of space vehicle navigation computations make the problem excessively complex for a course problem, we can run some low-dimensional, low-complexity examples to develop a better understanding of the machinery. We consider a two-dimensional state, linear-dynamics model, with a scalar, linear observation process. The dynamics correspond to a nominal ODE model of the form $\ddot{r} = -r$. Let

$$\zeta = \begin{pmatrix} r \\ v \end{pmatrix}$$

where $v = \dot{r}$. By the procedure from class, employing Euler’s method with time-step $\delta = 0.1$, this leads to Kalman filter dynamics

$$X_{k+1} = AX_k + BW_k, \quad \text{where} \quad A = \begin{pmatrix} 1 & \delta \\ -\delta & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Also, the $W_k$ are normal with mean, $m_w = 0$ and variance, $\sigma^2_w = \delta$ for each $k$. We let the observation process, which will occur once per time step, be given by

$$Y_k = HX_k + V_k, \quad \text{where} \quad H = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$ 

We suppose $V_k$ has mean, $m_v = 0$ and variance, $\sigma^2_v = 2$ for each $k$. Note that only the first component, the “$r$” component, of the state is being observed. Suppose the initial covariance of the state is given by

$$P_0 = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix}.$$ 

We would like to determine how well we can estimate the state of this system. Beginning with a dynamics update, propagate the covariance forward for 20 steps, yielding $M_k$ and $P_k$ for $k = 1$ to $k = 20$. (Specifically, assume that each time-step consists of a dynamics update followed by an observation update.) Plot the $(1, 1),$
(1, 2) and (2, 2) terms in $P_k$ as a function of $k$, preferably all in one plot window (although three separate plots are acceptable). Make sure that the curves are clearly labeled. Also, plot the (1, 1), (1, 2) and (2, 2) terms in $M_k$ as a function of $k$, preferably all in one plot window (although three separate plots are acceptable), again making sure that the curves are clearly labeled. Noting that the (1, 1) and (2, 2) terms in the matrices indicate the variance of $r$ and $v$, respectively. By approximately what factors would the standard deviations of the position and the velocity be reduced over this time period?

5. (5) Consider again the satellite of Problems 2 and 3. Let $t_k$ be the first time in your propagation/plots that the satellite comes above the local horizon of the groundstation. Let $\vec{r}^{eci}(t_k)$ be the nominal ECI position of the satellite at that time. Let the true/noise-impacted position be denoted by $\hat{\vec{r}}^{eci}(t_k)$. Let the difference be $X_k = \hat{\vec{r}}^{eci}(t_k) - \vec{r}^{eci}(t_k)$, which is the error in our knowledge of the satellite position. Suppose $X_k$ is a normal random vector with mean zero and covariance $M_k = \sigma_k^2 I$, where $I$ is a 3 by 3 identity matrix and $\sigma_k^2 = 0.0004 \text{ km}^2$. Note that $M_k$ is our a priori (pre-observation) covariance matrix. Suppose that at time $t_k$, we make a range measurement from the groundstation. The nominal measurement is

$$Z_k = g(\vec{r}^{eci}(t_k)) \doteq \left|\vec{r}^{eci}(t_k) - \vec{\rho}^{eci}(t_k)\right|,$$

where $\vec{\rho}^{eci}(t_k)$ is the position of the groundstation in ECI at time $t_k$, and for any $x$, $|x|$ denotes the length of the vector $x$. That is, $Z_k$ is the nominal distance from the groundstation to the satellite. The actual observation will take the form

$$\hat{Z}_k = g(\hat{\vec{r}}^{eci}(t_k)) + V_k,$$

where $V_k$ is a normal random variable with mean, $m_v = 0$ and variance, $\sigma_v^2 = 10^{-6} \text{ km}^2$. As usual, linearizing, we obtain

$$Y_k \doteq \hat{Z}_k - Z_k \simeq H_k X_k + V_k \doteq [\nabla g(\vec{r}^{eci}(t_k))]^T X_k + V_k.$$

What would the a posteriori (i.e., post-observation) covariance, $P_k$, of the satellite position-estimate error be?