Problem 1.

Solution.

$$\vec{R}^{\mathrm{ECI}} = \begin{pmatrix} 9000 \\ -5000 \\ 4000 \end{pmatrix} \text{km}.$$

1 hour has past since ECEF aligned with ECI. $\omega_e = 7.292\,115 \times 10^{-5}$ rad/s.

$$\vec{R}^{\mathrm{EF}} = G_3^{\omega_e \times 1\,\mathrm{hr}} \vec{R}^{\mathrm{ECI}} = \begin{pmatrix} 0.9657 & 0.2595 & 0 \\ -0.2595 & 0.9657 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix} \begin{pmatrix} 9000 \\ -5000 \\ 4000 \end{pmatrix} \text{km} = \begin{pmatrix} 7394.1 \\ -7164.3 \\ 4000 \end{pmatrix} \text{km}.$$

To convert from ECEF to local ENU coordinates, we use

$$\vec{R}^{\mathrm{ENU}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} G_2^{-\phi} G_3^{\lambda} (\vec{R}^{\mathrm{EF}} - \vec{R}_g^{\mathrm{EF}}),$$

where

$$\vec{R}_g^{\mathrm{EF}} = \frac{a}{\sqrt{1 - \tilde{e}^2 \sin^2 \phi}} \begin{pmatrix} \cos\phi \cos\lambda \\ \cos\phi \sin\lambda \\ (1 - \tilde{e}^2) \sin\phi \end{pmatrix} = \begin{pmatrix} -2445.7 \\ -4786.4 \\ 3422.1 \end{pmatrix} \text{km}.$$

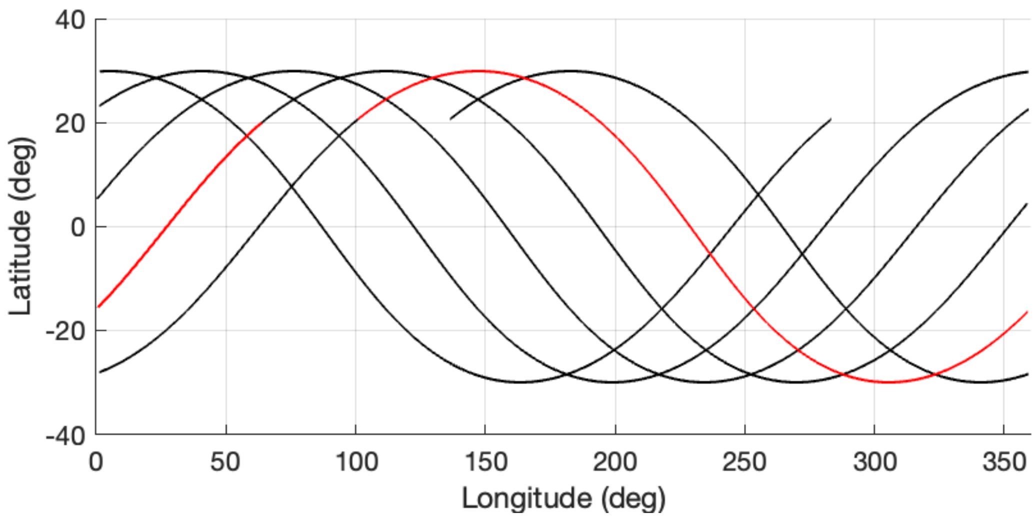
Hence

$$\vec{R}^{\mathrm{ENU}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} G_2^{-\phi} G_3^{\lambda} (\vec{R}^{\mathrm{EF}} - \vec{R}_g^{\mathrm{EF}}) = \begin{pmatrix} 9844.2 \\ 1759.9 \\ -1674.8 \end{pmatrix} \text{km}.$$

Since the last component ("U") is negative, it is not above local horizon.

Problem 2.

Solution.



Students must plot at least one period, match the general shape of the curve, and the plot must have correct bounds for latitude to receive full credits.

Students need not to plot multiple periods, and horizontal shift or offset of the curve does not matter.

Curves that have "gaps", like the colored part are correct.

Flat (horizontal) straight lines are plotting artifacts and can be ignored.

Sample code.

```
a = 9000;
e = 0.25;
incl = pi / 6;
Omega = pi / 3;
omega = pi / 4;
mu = 398600.4;
omega_e = 7.292115e-5;
nu = linspace(0, 10 * pi, 1001);
ecc_anomaly = 2 * atan(tan(nu / 2) * sqrt((1 - e) / (1 + e)));
mean_anomaly = ecc_anomaly - e * sin(ecc_anomaly);
wrap_around = [0, diff(mean_anomaly) < 0];
mean_anomaly = mean_anomaly + cumsum(wrap_around) * 2 * pi;
mean_motion = sqrt(mu / a^3);
time_since_peri = mean_anomaly / mean_motion;
time_since_ECEF_alignment = time_since_peri;
R_ECEF = zeros(3, 1001);
p = a * (1 - e^2);
for t = 1:size(R_ECEF, 2)
    R_peri = p / (1 + e * cos(nu(t))) * [cos(nu(t)); sin(nu(t)); 0];
    R_ECI = G3(Omega)' * G1(incl)' * G3(omega)' * R_peri;
    R_ECEF(:, t) = G3(omega_e * time_since_ECEF_alignment(t)) * R_ECI;
end
lat = asind(R_ECEF(3, :) ./ vecnorm(R_ECEF, 2, 1));
long = atan2d(R_ECEF(2, :), R_ECEF(1, :));
long(long < 0) = long(long < 0) + 360;
```

Problem 3.

Solution. Since the rocket starts with zero initial speed,
if terminal mass was 500 kg,

$$v = \Delta v = gI_{\mathrm{sp}} \ln \frac{5000\,\mathrm{kg}}{500\,\mathrm{kg}} = 6.7765\,\mathrm{km/s};$$

if terminal mass was 100 kg,

$$v = \Delta v = gI_{\mathrm{sp}} \ln \frac{5000\,\mathrm{kg}}{100\,\mathrm{kg}} = 11.5131\,\mathrm{km/s}.$$

Alternatively, using linear approximations
if terminal mass was 500 kg,

$$v = \Delta v = gI_{\mathrm{sp}} \ln \frac{4500\,\mathrm{kg}}{5000\,\mathrm{kg}} = 2.6487\,\mathrm{km/s};$$

if terminal mass was 100 kg,

$$v = \Delta v = gI_{\mathrm{sp}} \ln \frac{4900\,\mathrm{kg}}{5000\,\mathrm{kg}} = 2.8841\,\mathrm{km/s}.$$

Problem 4.

Solution. The burn must be performed at the nodes.
The true anomaly at the node is $\nu = -\omega$ (ascending) or $\pi - \omega$ (descending). For this problem, we may choose either $\nu = 0$ or $\nu = \pi$.
The velocity in perifocal coordinates at the nodes is

$$\vec{v}^{\text{peri}} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{pmatrix} -\sin\nu \\ e + \cos\nu \\ 0 \end{pmatrix} = \sqrt{\frac{\mu}{a}} \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} 0 \\ 3.2722 \\ 0 \end{pmatrix} \text{km/s}.$$

In the MCI coordinates, the velocity at the nodes (on the original orbit) is

$$\vec{v}^{\text{MCI}} = (G_3^{\Omega})^\top (G_1^{i})^\top (G_3^{\omega})^\top \vec{v}_{\text{peri}} = \pm \begin{pmatrix} -2.4541 \\ 1.4169 \\ 1.6361 \end{pmatrix} \text{km/s}.$$

After the inclination change, denoting the new inclination by $i' = 33°$, we have

$$\vec{v}^{\text{MCI}\prime} = (G_3^{\Omega})^\top (G_1^{i'})^\top (G_3^{\omega})^\top \vec{v}_{\text{peri}} = \pm \begin{pmatrix} -2.3766 \\ 1.3721 \\ 1.7821 \end{pmatrix} \text{km/s}.$$

Hence $\Delta\vec{v} = \pm \begin{pmatrix} 0.0775 \\ -0.0448 \\ 0.1461 \end{pmatrix}$ km/s (positive if at ascending node, negative otherwise) and $\Delta v = 0.171\,31\,\text{km/s}$.

$\boxed{\text{Alternatively, student may use } \Delta v = 2v\sin(\tfrac{\theta}{2}) = 0.171\,31\,\text{km/s or } \Delta v \approx v\theta = 0.171\,33\,\text{km/s.}}$

Using the exponential estimate,

$$\frac{\Delta m}{m(0)} = 1 - \exp(-\tfrac{\Delta v}{g I_{sp}}) = 5.5504\% \quad (5.5510\%).$$

Using linear estimate,

$$\frac{\Delta m}{m(0)} = \frac{\Delta v}{g I_{sp}} = 5.7103\% \quad (5.7110\%).$$