Problem 1.

Solution.

$$\begin{aligned}
\vec{R}^{\text{PERI}} &= G_3^{\omega} G_1^{i} G_3^{\Omega} \vec{R}^{\text{ECI}} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 29700 \\ 0 \\ 29700 \end{pmatrix} \text{km} \\
&\approx \begin{pmatrix} 41625 \\ 5250.3 \\ 1989.5 \end{pmatrix} \text{km.}
\end{aligned}$$

Problem 2.

Solution.

$$\begin{aligned}
\vec{R}^{\text{ECI}} &= (G_3^{\Omega})^{\top} (G_1^{i})^{\top} (G_3^{\omega})^{\top} \vec{R}^{\text{PERI}} \\
&= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\top} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^{\top} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\top} \begin{pmatrix} 29700 \\ 0 \\ 29700 \end{pmatrix} \text{km} \\
&\approx \begin{pmatrix} 10577 \\ -23682 \\ 33037 \end{pmatrix} \text{km.}
\end{aligned}$$

Problem 3.

Solution.
Mean motion $n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{42\,828.4\,\mathrm{km}^3/\mathrm{s}}{(6000\,\mathrm{km})^3}} \approx 4.4529 \times 10^{-4}\,\mathrm{rad/s}$.
Mean anomaly 2 hours after periapsis passage $M = n \times 2\,\mathrm{hours} \approx 3.2061\,\mathrm{rad}$.
Eccentric anomaly (Newton's iteration):

k	E_k
0	3.206 059 26
1	3.191 185 53
2	3.191 187 04

$E \approx 3.1912\,\mathrm{rad}$.
True anomaly $\nu = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) = -3.1052\,\mathrm{rad}$ (or $3.1780\,\mathrm{rad}$).
Distance to Mars $R = \frac{a(1-e^2)}{1+e\cos\nu} = 7797.8\,\mathrm{km}$.
Hence $\vec{R}^{\mathrm{peri}} = R\begin{pmatrix}\cos\nu\\ \sin\nu\\ 0\end{pmatrix} = \begin{pmatrix}-7792.6\\ -283.74\\ 0\end{pmatrix}$ km and $\vec{v}^{\mathrm{peri}} = \sqrt{\frac{\mu}{a(1-e^2)}}\begin{pmatrix}-\sin\nu\\ e+\cos\nu\\ 0\end{pmatrix} = \begin{pmatrix}-0.10191\\ -1.9586\\ 0\end{pmatrix}$ km/s.

$$\vec{R}^{\mathrm{MCI}} = (G_3^{\Omega})^\top (G_1^{i})^\top (G_3^{\omega})^\top \vec{R}^{\mathrm{peri}} = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}^\top \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}^\top \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{bmatrix}^\top \begin{pmatrix}-7792.6\\ -283.74\\ 0\end{pmatrix}\,\mathrm{km}$$

$$= \begin{pmatrix}1628.4\\ -7071.1\\ -2855.4\end{pmatrix}\,\mathrm{km}$$

$$\vec{v}^{\mathrm{MCI}} = (G_3^{\Omega})^\top (G_1^{i})^\top (G_3^{\omega})^\top \vec{v}^{\mathrm{peri}} = \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}^\top \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}^\top \begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\ 0 & 0 & 1\end{bmatrix}^\top \begin{pmatrix}-0.10191\\ -1.9586\\ 0\end{pmatrix}\,\mathrm{km/s}$$

$$= \begin{pmatrix}1.7132\\ 0.69332\\ -0.65646\end{pmatrix}\,\mathrm{km/s}.$$

Problem 4.

Solution. $\mu_{\text{Mars}} = 42\,828.4\,\text{km}^3/\text{s}^2$.
$R = |\vec{R}^{\text{MCI}}| = 3469.4\,\text{km}$.
Specific energy $\mathcal{E} = \frac{v}{2} - \frac{\mu_{\text{Mars}}}{R} = -4.1459\,\text{MJ/kg}$.

$\boxed{\text{Semi-major axis (vis-viva equation) } a = -\frac{\mu}{2\mathcal{E}} = 5165.2\,\text{km}.}$

Specific angular momentum $\vec{h}^{\text{MCI}} = \vec{R} \times \vec{v} = \begin{pmatrix} 7019.3 \\ 1.3220 \\ 12163 \end{pmatrix} \text{km}^2/\text{s}$.

Unit vector in the direction of $\vec{h}$: $\vec{u}_h^{\text{MCI}} = \begin{pmatrix} 0.49983 \\ 0.00009 \\ 0.86612 \end{pmatrix}$.

"Parameter" $p = \frac{h^2}{\mu} = 4604.8\,\text{km}$.

$\boxed{\text{Eccentricity } e = \sqrt{1 - \frac{p}{a}} = 0.32938.}$

$\boxed{\text{Inclination } i = \arccos[\vec{u}_h^{\text{MCI}}]_3 = 0.523\,40\,\text{rad}.}$

$\sin\Omega = \frac{[\vec{u}_h^{\text{MCI}}]_1}{\sin i} \approx 1.0000$, $\cos\Omega = -\frac{[\vec{u}_h^{\text{MCI}}]_2}{\sin i} \approx 0.00019$.

$\boxed{\text{Longitude of node } \Omega \approx \frac{\pi}{2}\text{rad.}}$

$\cos\nu(0) = \frac{p/R-1}{e} = 0.99357$.
We check that $\vec{R} \cdot \vec{v} \approx 394 > 0$.
Therefore, $\boxed{\nu(0) = \arccos\cos\nu(0) = 0.113\,48\,\text{rad}.}$

$\hat{\hat{\vec{R}}} = G_1^i G_3^\Omega \vec{R}^{\text{MCI}} = \begin{pmatrix} -3337.8 \\ 946.32 \\ 0 \end{pmatrix} \text{km}$.

$\cos\theta = \frac{[\hat{\hat{\vec{R}}}]_1}{R} \approx -0.96208, \sin\theta = \frac{[\hat{\hat{\vec{R}}}]_2}{R} \approx -0.27276$.
Therefore, $\theta = \arctan\frac{-0.96208}{-0.27276} \pm \pi = -2.8653\,\text{rad}$ or $3.4179\,\text{rad}$.

$\boxed{\text{Argument of periapsis } \omega = \theta - \nu = -2.9288\,\text{rad or } 3.3044\,\text{rad}.}$

Eccentric anomaly $E(0) = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right) = 0.080\,65\,\text{rad}$.
Mean anomaly $M(0) = E(0) - e\sin(E(0)) = 0.054\,11\,\text{rad}$.
Mean motion $n = \sqrt{\frac{\mu}{a^3}} = 5.5749 \times 10^{-4}\,\text{rad/s}$

$\boxed{\tau(0) = \frac{-M}{n} = -97.062\,\text{s}}$ or ($11\,173.4\,\text{s}$, next passage).

Periapsis $r_p = a(1-e) = 3463.9\,\text{km} > r_{\text{Mars,eq}} \approx 3396\,\text{km}$. Therefore, the satellite is not in danger of crashing into Mars.

Using any radius for Mars is acceptable.

Problem 5.

Solution. 1 hour has passed after alignment.
Angular rate of Earth $\omega_e = 7.292\,115 \times 10^{-5}$ rad/s.

$$\begin{aligned}
\vec{R}^{\text{EF}} &= G_3^{\omega_e \times 1\,\text{hr}} \vec{R}^{\text{ECI}} = G_3^{0.262516} \begin{pmatrix} 8000 \\ -4000 \\ 3000 \end{pmatrix} \text{km} \\
&= \begin{pmatrix} 0.96574 & 0.25951 & 0 \\ -0.25951 & 0.96574 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8000 \\ -4000 \\ 3000 \end{pmatrix} \text{km} \\
&= \begin{pmatrix} 6687.9 \\ -5939.1 \\ 3000 \end{pmatrix} \text{km.}
\end{aligned}$$