Problem 1.

Solution.
Write $\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{v}) = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r_2 v_3 - r_3 v_2 \\ r_3 v_1 - r_1 v_3 \\ r_1 v_2 - r_2 v_1 \end{pmatrix} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix} r_2 v_3 \\ r_3 v_1 \\ r_1 v_2 \end{pmatrix} - \begin{pmatrix} r_3 v_2 \\ r_1 v_3 \\ r_2 v_1 \end{pmatrix}\right] \\
&= \begin{pmatrix} \dot{r}_2 v_3 \\ \dot{r}_3 v_1 \\ \dot{r}_1 v_2 \end{pmatrix} + \begin{pmatrix} r_2 \dot{v}_3 \\ r_3 \dot{v}_1 \\ r_1 \dot{v}_2 \end{pmatrix} - \begin{pmatrix} \dot{r}_3 v_2 \\ \dot{r}_1 v_3 \\ \dot{r}_2 v_1 \end{pmatrix} - \begin{pmatrix} r_3 \dot{v}_2 \\ r_1 \dot{v}_3 \\ r_2 \dot{v}_1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \dot{r}_2 v_3 \\ \dot{r}_3 v_1 \\ \dot{r}_1 v_2 \end{pmatrix} - \begin{pmatrix} \dot{r}_3 v_2 \\ \dot{r}_1 v_3 \\ \dot{r}_2 v_1 \end{pmatrix}} + \underbrace{\begin{pmatrix} r_2 \dot{v}_3 \\ r_3 \dot{v}_1 \\ r_1 \dot{v}_2 \end{pmatrix} - \begin{pmatrix} r_3 \dot{v}_2 \\ r_1 \dot{v}_3 \\ r_2 \dot{v}_1 \end{pmatrix}} \\
&= \qquad\quad \dot{\vec{r}} \times \vec{v} \qquad\quad + \qquad\quad \vec{r} \times \dot{\vec{v}}.
\end{aligned}$$

Problem 2.

Solution. Consider $\vec{h} = \vec{r} \times \vec{v}$. We see that

$$\dot{\vec{h}} = \dot{\vec{r}} \times \vec{v} + \vec{r} \times \dot{\vec{v}}.$$

But since $\dot{\vec{r}} = \vec{v}$, we simply have

$$\dot{\vec{h}} = \vec{r} \times \ddot{r}.$$

Case 1:

$$\dot{\vec{h}} = \vec{r} \times \left(-c_1 \frac{e^{-|\vec{r}|} + 1}{|\vec{r}|^3}\vec{r}\right) = -c_1 \frac{e^{-|\vec{r}|} + 1}{|\vec{r}|^3} \cancel{\vec{r} \times \vec{r}} = 0.$$

We conclude that $\vec{h}$ is constant, and in particular, its direction does not change. But since $\vec{h}$ is always orthogonal to both $\vec{r}$ and $\vec{v} = \dot{\vec{r}}$, it follows that the the relative motion of the two bodies stays in the same plane.
Case 2:

$$\dot{\vec{h}} = \vec{r} \times \left(-c_1 \frac{e^{-|\vec{r}|} + 1}{|\vec{r}|^3}\vec{r} - c_2 \dot{\vec{r}}\right) = -c_1 \frac{e^{-|\vec{r}|} + 1}{|\vec{r}|^3} \cancel{\vec{r} \times \vec{r}} - c_2 \vec{r} \times \dot{\vec{r}} = -c_2 \vec{r} \times \dot{\vec{r}}.$$

We note that $\vec{r} \times \dot{\vec{r}} = \vec{h}$. Hence $\dot{\vec{h}} = -c_2 \vec{h}$. Despite that $\dot{\vec{h}} \neq 0$, the change of $\vec{h}$ is always parallel to $\vec{h}$ itself. Hence the direction of $\vec{h}$ does not change. Again, since $\vec{h}$ is orthogonal to both $\vec{r}$ and $\vec{v} = \dot{\vec{r}}$, the the relative motion of the two bodies stays in the same plane.

Problem 3.

Recall that specific energy is an integral of motion. The specific energy can firstly be computed using $r = 6000\,\mathrm{km}$ and $v = 5.6\,\mathrm{km/s}$, yielding

$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu_{\mathrm{Mars}}}{r} \approx 8.5419\,\mathrm{km}^2/\mathrm{s}^2.$$

Then at $r = 8000\,\mathrm{km}$, we may solve for v again, getting

$$v = \sqrt{2\left(\mathcal{E} + \frac{\mu_{\mathrm{Mars}}}{r}\right)} = \sqrt{2\left(8.5419\,\mathrm{km}^2/\mathrm{s}^2 + \frac{42\,828.4\,\mathrm{km}^3/\mathrm{s}^2}{8000\,\mathrm{km}}\right)} \approx 5.2717\,\mathrm{km/s}.$$