Last lecture

For any two-body problem,

$$\vec{h} = \vec{R} \times \vec{v},$$
$$\mathcal{E} = \frac{1}{2}v^2 - \frac{\mu}{R}$$

are constant.

Coordinate system

We consider a cylindrical coordinate system whose third (i.e. z-) axis points in the same direction as $\vec{h}$. Recall that $\vec{h}$ is orthogonal to both $\vec{R}$ and $\vec{v}$. This implies that the trajectory of $\vec{R}$ is planar, i.e. there is no motion along the z-axis.
We write

$$\vec{R}(t) = R(t)\begin{pmatrix}\cos(\theta(t))\\ \sin(\theta(t))\\ 0\end{pmatrix} \qquad \rightsquigarrow \qquad \vec{v} = \begin{pmatrix}\dot{R}\cos(\theta) - R\sin(\theta)\dot{\theta}\\ \dot{R}\sin(\theta) + R\cos(\theta)\dot{\theta}\\ 0\end{pmatrix}.$$

Angular momentum revisited

Observe then

$$\vec{h} = \vec{R} \times \vec{v}$$
$$= \begin{pmatrix} 0 \\ 0 \\ R\cos(\theta)(\dot{R}\sin(\theta) + R\cos(\theta)\dot{\theta}) - R\sin(\theta)(\dot{R}\cos(\theta) - R\sin(\theta)\dot{\theta}) \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ R^2\dot{\theta} \end{pmatrix}.$$

This implies that $|\vec{h}| = R^2\dot{\theta}$. (Note in particular that $\frac{\mathrm{d}}{\mathrm{d}t}(R^2\dot{\theta}) = 2R\dot{R}\dot{\theta} + R^2\ddot{\theta} = R(2\dot{R}\dot{\theta} + R\ddot{\theta}) = 0$.)

Acceleration due to gravitation

Let $\hat{I}^1(\theta(t)) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \\ 0 \end{pmatrix}$ and $\hat{I}^2(\theta(t)) = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \\ 0 \end{pmatrix}$.

Then $\vec{R} = R\hat{I}^1$.

$$\begin{aligned} \vec{a} = \ddot{\vec{R}} &= \ddot{R}\hat{I}^1 + 2\dot{R}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{I}^1\right]\dot{\theta} + R\left(\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\hat{I}^1\right]\dot{\theta}^2 + \left[\frac{\mathrm{d}}{\mathrm{d}\theta}\hat{I}^1\right]\ddot{\theta}\right) \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{I}^1 + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{I}^2. \end{aligned}$$

whereas by the formula for gravitational force

$$\vec{a} = -\frac{\mu}{R^2}\hat{I}^1.$$

Equating these two expressions, we get the following equations:

$$\ddot{R} - R\dot{\theta}^2 + \frac{\mu}{R^2} = 0, \qquad 2\dot{R}\dot{\theta} + R\ddot{\theta} = 0.$$

Shape of the Orbit

If we think of R as a function of θ (and θ is in turn a function of time), then the equation above may be rewritten as

$$\underbrace{\left[\frac{\mathrm{d}^2 R}{\mathrm{d}\theta^2}\right]\dot{\theta}^2 + \left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]\overbrace{\left(-\frac{2\dot{R}\dot{\theta}}{R}\right)}^{\ddot{\theta}}}_{\ddot{R}} - R\dot{\theta}^2 + \frac{\mu}{R^2} = 0.$$

Dividing both sides by $\dot{\theta}^2$, we have

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\theta^2} - \frac{2}{R}\left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]\frac{\dot{R}}{\dot{\theta}} - R = -\frac{\mu}{R^2\dot{\theta}^2}.$$

Dividing again by R^2, we arrive at

$$\frac{1}{R^2}\left[\frac{\mathrm{d}^2 R}{\mathrm{d}\theta^2} - \frac{2}{R}\left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]^2 - R\right] = -\frac{\mu}{h^2}.$$

The solution to the ODE above is of the form

$$R(\theta) = \frac{p}{1 + e\cos(\theta - \omega)}, \qquad (\dagger)$$

where $p, e \geq 0$, $\omega \in [0, 2\pi)$.

Eccentricity

The constant e in (†) is known as the "eccentricity". If $e = 0$, the orbit is a circle. If $e \in (0, 1)$, the orbit is an ellipse. The origin of the coordinate system (i.e. the center of the planet) is the focus (not the center) of the ellipse.

By (†), for $e \in [0, 1)$, R reaches its minimum when $\theta - \omega = 0$ and R reaches its maximum when $\theta - \omega = \pi$. Hence the semi-major axis a of the orbit is given by

$$2a = R|_{\theta-\omega=0} + R|_{\theta-\omega=\pi} = \frac{p}{1+e} + \frac{p}{1-e} \quad \rightsquigarrow \quad a = \frac{p}{1-e^2}.$$