Two-body problem

Consider two spherically symmetric objects with mass m_1 and m_2 and position $\vec{x}^1$ and $\vec{x}^2$ respectively, interacting with each other via gravity. The potential energy of the system is given by

$$-\frac{Gm_1 m_2}{|\vec{x}^2 - \vec{x}^1|}.$$

Equation of motion

The equation of motion for each object is given by

$$m_1 \ddot{\vec{x}}^1 = \vec{F}^1 = -\frac{G m_1 m_2}{|\vec{x}^2 - \vec{x}^1|^3}(\vec{x}^1 - \vec{x}^2)$$
$$m_2 \ddot{\vec{x}}^2 = \vec{F}^2 = -\frac{G m_1 m_2}{|\vec{x}^2 - \vec{x}^1|^3}(\vec{x}^2 - \vec{x}^1).$$

Dividing by mass, we obtain

$$\ddot{\vec{x}}^1 = -\frac{G m_2}{|\vec{x}^2 - \vec{x}^1|^3}(\vec{x}^1 - \vec{x}^2)$$
$$\ddot{\vec{x}}^2 = -\frac{G m_1}{|\vec{x}^2 - \vec{x}^1|^3}(\vec{x}^2 - \vec{x}^1).$$

Center of mass and relative position

Instead of $\vec{x}^1, \vec{x}^2$, we may also describe the system by its center of mass and relative position from 1 to 2.
The center of mass of the system is given by

$$\vec{x}^c = \frac{m_2}{m_1 + m_2}\vec{x}^1 + \frac{m_1}{m_1 + m_2}\vec{x}^2.$$

The relative position is given by

$$\vec{R} = \vec{x}^2 - \vec{x}^1.$$

EOM for center of mass

$$\ddot{\vec{x}}^c = \frac{m_2}{m_1 + m_2}\ddot{\vec{x}}^1 + \frac{m_1}{m_1 + m_2}\ddot{\vec{x}}^2 = 0.$$

We conclude that the center of mass moves in a straight line; that is,

$$\vec{x}^c(t) = \vec{x}^c(0) + \vec{v}^c(0)t, \qquad \vec{v}^c(0) = \frac{m_2}{m_1 + m_2}\vec{v}^1(0) + \frac{m_1}{m_1 + m_2}\vec{v}^2(0).$$

EOM for relative position

$$\ddot{\vec{R}} = -\frac{Gm_1}{|\vec{R}|^3}\vec{R} - \frac{Gm_2}{|\vec{R^3}|}\vec{R}$$
$$= -\frac{G(m_1 + m_2)}{|\vec{R}|^3}\vec{R}. \qquad (*)$$

In the case where object 1 is the Earth and object 2 is a spacecraft, we may neglect the mass of the spacecraft and wrirte the equation above as

$$\ddot{\vec{R}} \approx -\frac{G\mu_e}{|\vec{R}|^3}\vec{R}.$$

where $\mu_e = Gm_e \approx 398\,600.4\,\mathrm{km}^3/\mathrm{s}^2$.

Specific angular momentum

The "specific angular momentum" is defined as $\vec{h} = \vec{R} \times \vec{v}$ (where $\vec{v} = \dot{\vec{R}}$).
Observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{h} = \vec{R} \times \dot{\vec{v}} + \dot{\vec{R}} \times \vec{v} = \vec{R} \times \ddot{\vec{R}} + \dot{\vec{R}} \times \dot{\vec{R}}.$$

Since $\vec{a} \times (k\vec{a}) = \vec{0}$ for all scalars k and all vectors $\vec{a}$, by $(*)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{h} = 0 \qquad \rightsquigarrow \vec{h} = \vec{R} \times \vec{v} \text{ is constant.}$$

By properties of cross product, $\vec{h}$ is always orthogonal to $\vec{R}$ and $\vec{v}$. That is, $\vec{R}$ and $\vec{v}$ lie in the plane whose normal vector is (parallel to) $\vec{h}$.
Let $\vec{u}_h$ denote the unit vector in the direction of $\vec{h}$. $\vec{u}_h$ defines the plane in which the spacecraft moves. Let h denote the magnitude of $\vec{h}$. h (partially) defines how the spacecraft moves in the plane.

Specific energy

Multiplying $(*)$ by $\vec{v}$ (dot product), we have

$$0 = \ddot{\vec{R}} \cdot \vec{v} + \frac{\mu}{|\vec{R}|^3} \vec{R} \cdot \vec{v}.$$

Indeed, the RHS may be rewritten as

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\vec{v} \cdot \vec{v}}{2} - \frac{\mu}{|\vec{R}|} \right].$$

We see that $\mathcal{E} = \frac{|\vec{v}|^2}{2} - \frac{\mu}{|\vec{R}|}$ is constant. This is known as the "specific energy".