Last lecture

Rotation matrices
Coordinate transforms

Solving for trajectory from orbital elements

Given $a, e, i, \Omega, \omega, \tau$ and some time t, we can find $\vec{R}^{\mathrm{ECI}}$ and $\vec{v}^{\mathrm{ECI}}$ as follows.

1. Find mean motion $n = \sqrt{\frac{\mu}{a^3}}$;
2. Find mean anomaly $M = n(t - \tau)$;
3. Find eccentric anomaly by solving $M = E - e\sin(E)$;
4. Find true anomaly ν by solving $\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$;
5. Find $R = \frac{a(1-e^2)}{1+e\cos\nu}$;
6. $\vec{R}^{\mathrm{peri}} = R\begin{pmatrix}\cos\nu \\ \sin\nu \\ 0\end{pmatrix}$;
7. $\vec{v}^{\mathrm{peri}} = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1}{1-e^2}}\begin{pmatrix}-\sin\nu \\ e+\cos\nu \\ 0\end{pmatrix}$;
8. $\vec{R}^{\mathrm{ECI}} = (G_3^{\Omega})^\top (G_1^{i})^\top (G_3^{\omega})^\top \vec{R}^{\mathrm{peri}}$;
9. $\vec{v}^{\mathrm{ECI}} = (G_3^{\Omega})^\top (G_1^{i})^\top (G_3^{\omega})^\top \vec{v}^{\mathrm{peri}}$.

Finding orbital elements I

Suppose instead that we know $\vec{R}^{\mathrm{ECI}}$ and $\vec{v}^{\mathrm{ECI}}$ at some time t, and we'd like to find $a, e, i, \Omega, \omega, \tau$.
Denote $R = |\vec{R}^{\mathrm{ECI}}|$ and $v = |\vec{v}^{\mathrm{ECI}}|$. (Note that magnitude of vectors does not depend on the underlying coordinate system.)

1. Find specific energy $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{R}$; [Verify $\mathcal{E} < 0$ before proceeding]
2. Find semi-major axis $a = -\frac{\mu}{2\mathcal{E}}$.
3. Find specific angular momentum $\vec{h}^{\mathrm{ECI}} = \vec{r}^{\mathrm{ECI}} \times \vec{v}^{\mathrm{ECI}}$;
4. Find "parameter" $p = \frac{|\vec{h}^{\mathrm{ECI}}|^2}{\mu}$;
5. Find eccentricity $e = \sqrt{1 - \frac{p}{a}}$;

Finding orbital elements II

6. Find inclination $i \in [0, \pi]$ and longitude of node $\Omega \in [0, 2\pi)$ using $\vec{u}_h^{\mathrm{ECI}} = \begin{pmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{pmatrix}$
$i = \arccos([\vec{u}_h^{\mathrm{ECI}}]_3)$; use atan2 or arctan together with the sign of first two components to find Ω);
7. Find current true anomaly:
let $k = \cos(\nu) = \frac{1}{e}\left[\frac{a(1-e^2)}{R} - 1\right]$
if $\vec{R}^{\mathrm{ECI}} \cdot \vec{v}^{\mathrm{ECI}} > 0$, $\nu = \arccos(k)$
if $\vec{R}^{\mathrm{ECI}} \cdot \vec{v}^{\mathrm{ECI}} < 0$, $\nu = -\arccos(k)$
if $\vec{R}^{\mathrm{ECI}} \cdot \vec{v}^{\mathrm{ECI}} = 0$, compare R with a to determine whether it's at periapsis or apoapsis;

Finding orbital elements III

8. Find argument of periapsis ω: let $\hat{\hat{\vec{R}}} = G_1^i G_3^\Omega \vec{R}^{\mathrm{ECI}}$; use $\cos(\theta) = \frac{[\hat{\hat{\vec{R}}}]_1}{R}$ and $\sin(\theta) = \frac{[\hat{\hat{\vec{R}}}]_2}{R}$ to find θ; $\omega = \theta - \nu$.
9. Find time of periapsis passage τ:
   compute eccentric anomaly $\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2}$
   compute mean anomaly $M = E - e \sin(E)$
   $\tau = -\frac{M}{n}$.

Earth-centered earth-fixed (ECEF) system

ECEF is a coordinate system that “rotates with the Earth”, in which:

- the origin is at the center of Earth;
- I_1^{EF} is in the equatorial plane;
- I_2^{EF} is also in the equatorial plane, 90° apart from (east of) I_1^{EF};
- I_3^{EF} is the same as I_3^{ECI}.

We often specify ECEF by saying “ECEF and ECI aligned at some time t_0”. This means that at time t, $\vec{R}^{\mathrm{EF}} = G_3^{\omega_e(t-t_0)}\vec{R}^{\mathrm{ECI}}$ (ω_e denotes the average rotation rate of the earth).

Sidereal day

Due to the rotation of the Earth, one solar day (i.e. day measured as "noon to noon") is longer than one sidereal day (i.e. day measured as Earth rotates 360°).
One sidereal day is $86\,164.091\,\mathrm{s}$.