Last lecture

- True anomaly as a function of time.
- Position and velocity formulae.

Vis-viva equation

From the velocity equation, we obtain

$$
\begin{aligned}
|\vec{v}|^2 &= \frac{\mu}{a(1-e^2)}(\sin^2(\nu) + (e+\cos(\nu))^2) \\
&= \frac{mu}{a(1-e)^2}(\sin^2(\nu)^2 + \cos^2(\nu) + 2e\cos(\nu) + e^2) \\
&= \frac{mu}{a(1-e)^2}(2(1+e\cos(\nu)) + e^2 - 1) = \frac{2\mu(1+e\cos(\nu))}{a(1-e^2)} - \frac{\mu}{a} \\
&= \frac{2\mu}{R} - \frac{\mu}{a}.
\end{aligned}
$$

That is, $\frac{v^2}{2} = \frac{\mu}{R} - \frac{\mu}{2a}$. More generally, for non-elliptical orbits, we still have

$$
\frac{v^2}{2} = \frac{\mu}{R} - \frac{\mu^2(1-e^2)}{2h^2}, \text{ or } \boxed{\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{R} = -\frac{\mu^2(1-e^2)}{2h^2}}.
$$

Eccentricity

From the boxed equation in the previous slide, we see that

- $\mathcal{E} < 0 \iff e \in [0, 1)$ (elliptic), and in this case $R < -\frac{\mu}{\mathcal{E}}$;
- If $\mathcal{E} = 0 \iff e = 1$ (parabolic), and in this case $R \to \infty$ as $\nu \to \pm\pi$;
- If $\mathcal{E} > 0 \iff e > 1$ (hyperbolic), and in this case $R \to \infty$ as $\nu \to \pm \arccos(-e^{-1})$.

The escape speed is therefore given by solving $\mathcal{E} = 0$ for v, yielding

$$v = \sqrt{\frac{2\mu}{R}}.$$

Earth-centered Inertial (ECI) system

ECI is a right-handed coordinate system centered at the Earth. We refer to its 3 coordinate axes as I^1, I^2 and I^3 axes (in lieu of x, y, z). Its I^3-axis is aligned with the rotation axis of the Earth (i.e. from the south pole to the north pole).

For a given orbit, the intersection of the orbit plane and the I^1-I^2 plane is known as the line of nodes. The angle between the I^1 axis and the line of nodes is the longitude of node Ω. The (dihedral) angle from the I^1-I^2 plane to the orbit plane is the inclination i.

We rotate the ECI axes by Ω about the I^3 axis first and then by i about the i axis, thus obtaining a new coordinate system $\hat{\hat{I}}^{1,2,3}$ in which the orbit plane coincides with the $\hat{\hat{I}}^1$-$\hat{\hat{I}}^2$ plane and the angular momentum is in the direction of $\hat{\hat{I}}^3$.

In the $\hat{\hat{I}}^1$-$\hat{\hat{I}}^2$ plane, the angle from $\hat{\hat{I}}^1$ to the line from the origin is the argument of periapsis ω. Hence lastly, we rotate about $\hat{\hat{I}}^3$ axis by ω, so that periapsis is on (the positive) $\hat{\hat{I}}^1$ axis.