Last lecture

Orbital period $T = \frac{A}{\dot{A}} = \frac{\pi a^2 \sqrt{1-e^2}}{h/2} = \frac{2\pi a^2 \sqrt{1-e^2}}{h} = 2\pi \sqrt{\frac{a^3}{\mu}}$.

Mean motion $n = \sqrt{\frac{\mu}{a^3}}$.

Mean anomaly and true anomaly

Rate of change of true anomaly is given by $\dot{\nu} = \frac{h}{R^2}$.
Mean anomaly M at time t is defined to be $n(t - \tau)$, where n is mean motion, and τ is the time when the spacecraft passes the periapsis.
Since an elliptic orbit is symmetric about its "axis", the mean anomaly and true anomaly matches at periapsis and apoapsis.

Position as function of time

Suppose a, e, τ are known. We'd like to solve for the position of the spacecraft as a function of time.

1. Find mean motion $n = \sqrt{\frac{a^3}{\mu}}$;
2. Find mean anomaly $M = M(t - \tau)$ at time t;
3. Find eccentric anomaly E, by solving $E - e\sin(E) = M$;
4. Find true anomaly at is given by $\tan\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$;
5. Find $R = \frac{p}{1+e\cos\nu} = \frac{a(1-e^2)}{1+e\cos\nu}$;
6. $\vec{R} = R\begin{pmatrix}\cos\nu\\ \sin\nu\\ 0\end{pmatrix}$.

Eccentric anomaly

In order to find E, we need to solve the equation $E - e\sin(E) = M$, for which we may use the Newton's method. Starting with the initial guess $E_0 = M$, and find

$$E_{k+1} = E_k - \frac{E_k - e\sin(E_k) - M}{1 - e\cos(E_k)},$$

for $k = 0, 1, \cdots$.
Then E_k converges to the solution to $E - e\sin(E) = M$.

Velocity as a function of time

Differentiating $\vec{R}$, we get

$$\vec{v} = \dot{R} \begin{pmatrix} \cos\nu \\ \sin\nu \\ 0 \end{pmatrix} + R \begin{pmatrix} -\sin\nu \\ \cos\nu \\ 0 \end{pmatrix} \dot{\nu},$$

in which

$$\dot{R} = \frac{a(1-e^2)}{(1+e\cos\nu)^2} e\sin(\nu)\dot{\nu}, \quad \dot{\nu} = \frac{h}{R^2} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1+e\cos\nu)^2.$$

Substituting and simplifying, we obtain

$$\vec{v} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{pmatrix} -\sin\nu \\ e+\cos\nu \\ 0 \end{pmatrix}.$$

Velocity at periapsis and apoapsis

Periapsis

$$\vec{v}_{\mathrm{peri}} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{pmatrix} 0 \\ 1+e \\ 0 \end{pmatrix}, v_{\mathrm{peri}} = \sqrt{\frac{\mu}{a}\frac{1+e}{1-e}}.$$

Apoapsis

$$\vec{v}_{\mathrm{apo}} = \sqrt{\frac{\mu}{a(1-e^2)}} \begin{pmatrix} 0 \\ e-1 \\ 0 \end{pmatrix}, v_{\mathrm{apo}} = \sqrt{\frac{\mu}{a}\frac{1-e}{1+e}}.$$