Aerobraking

Consider a spacecraft in a high-energy orbit, and we'd like to slow it down using atmosphere.

Suppose the spacecraft is an elliptical orbit, the periapsis of which is inside the atmosphere. In a highly elliptical orbit, aerobraking happens only near the periapsis, and it affects mostly the apoapsis.

The drag (acceleration) due to atmosphere depends on the area S of the spacecraft, coefficient of drag C_D, and atmospheric density ρ, etc. We use

$$a_D = \frac{C_D S \rho v^2}{2m}$$

for this class.

Aerobraking

Using vis-viva equation, the speed as a function of true anomaly is given by

$$v = \sqrt{\frac{\mu}{R_p} \frac{1 + 2e\cos\nu + e^2}{1+e}} \overset{\text{near peri}}{\approx} \sqrt{\frac{\mu}{R_p}} \sqrt{1+e}.$$

As an approximation, $\dot{\mathcal{E}} = v\dot{v} = -va_D = -\frac{C_D S \rho}{2m} v^3$. Hence

$$a = -\frac{\mu}{2\mathcal{E}}, \quad \dot{a} = -\frac{\mu}{2\mathcal{E}^2}\dot{\mathcal{E}}.$$

Atmospheric density

$$\rho = \rho_0 \exp\left(-\frac{R - R_0}{h}\right),$$

where ρ_0 is the density at a reference altitude R_0, and h is the scale height (constant).

Apoapsis change

For each pass, the change of R_a is given by (noting that $R_a = 2a - R_p$),

$$\Delta R_a = \int_{-\tau}^{\tau} \dot{R}_a \, \mathrm{d}t = \int_{-\tau}^{\tau} 2\dot{a} \, \mathrm{d}t = \int_{-\tau}^{\tau} -\frac{\mu}{\mathcal{E}^2} \dot{\mathcal{E}} \, \mathrm{d}t,$$

where $\dot{\mathcal{E}}$ is given on slide #2.

Hill's or Clohessy-Wiltshire Equation

Consider two spacecraft relatively close to each other. Denote their positions by $\vec{R}$ and $\vec{R}^*$. Consider the relative position
$\vec{\Delta}_R = \ddot{\vec{R}} - \ddot{\vec{R}}^*$.
The acceleration at $\vec{R}$ is given by $\vec{g}(\vec{R}) = -\frac{\mu}{|\vec{R}|^3}\vec{R}$. Then

$$\vec{\Delta}_R = g(\vec{R}) - g(\vec{R}^*) \approx \frac{\mathrm{d}\vec{g}}{\mathrm{d}\vec{R}}(\vec{R}^*)[\vec{R} - \vec{R}^*],$$

where $\frac{\mathrm{d}\vec{g}}{\mathrm{d}\vec{R}}$ is the Jacobian matrix of $\vec{g}$.
Recall that the gravitational acceleration $\vec{g}$ is derived from the gravitational potential $-\frac{\mu}{R}$. Indeed, the i, j-th component of $\frac{\mathrm{d}\vec{g}}{\mathrm{d}\vec{R}}$ is

$$\left[\frac{\mathrm{d}\vec{g}}{\mathrm{d}\vec{R}}(\vec{R}^*)\right]_{i,j} = \frac{\mu}{|\vec{R}^*|^5}[3\vec{R}^*\vec{R}^{*\top} - |\vec{R}^*|^2 I].$$

For example, if $\vec{R}^*(t) = (R^*(t), 0, 0)$. Then

$$\ddot{\vec{\Delta}}_R \approx \frac{\mu}{|\vec{R}^*|^3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \vec{\Delta}_R.$$

For two spacecrafts moving in circular orbits, in a coordinate system moving with $\vec{R}^*$, whose coordinate axes are aligned with the radial, tangential and out-of-plane direction of $\vec{R}^*$. Write the components of relative position and velocity in this system as $\hat{\Delta} = (x, y, z)$ and $\dot{\hat{\Delta}} = (u, v, w)$. Then the Clohessy-Wiltshire equation is a system of linear ODEs that models the relative motion in this coordinate:

$$\begin{aligned} \dot{u} &= 3n^2 x + 2nv \\ \dot{v} &= -2nu \\ \dot{w} &= -n^2 z \\ \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= w. \end{aligned}$$