Chemical Propulsion (Burn)

Consider a vehicle moving in a straight line running its rocket engine with nozzle velocity v_e.

Assume at initial time, the vehicle has mass m and velocity v, and at time t, the vehicle has mass $m - \Delta m$ and is now moving with velocity $v + \Delta v$.

By conservation of momentum,

$$(m - \Delta m)(v + \Delta v) + \Delta m(v - v_e) = mv,$$

which yields

$$mv + m\Delta v - \Delta mv - \underbrace{\Delta m\Delta v}_{\approx 0} + \Delta m(v - v_e) = mv.$$

This implies $m\Delta v - \Delta m v_e = 0$. Rearranging, this implies[1]

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -v_e\frac{\mathrm{d}m}{\mathrm{d}t} \quad \rightsquigarrow \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v_e}{m}\frac{\mathrm{d}m}{\mathrm{d}t}.$$

[1] Δm is the decrease in m.

Integrating the equation, we get

$$v(t) - v(0) = v_e \ln \frac{m(0)}{m(t)}.$$

Note that, theoretically, as $m(t) \downarrow 0$, $v(t) \uparrow \infty$.
In 3-D, suppose the burn is in the direction (of a unit vector) $\vec{u}$; we similarly have

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{v_e}{m}\frac{\mathrm{d}m}{\mathrm{d}t}\vec{u}.$$

v_e is usually written as gI_{sp} where g is the standard gravitational acceleration on Earth and I_{sp} is the specific impulse (a property of rocket engine). A typical value for gI_{sp} is $3\,\mathrm{km/s}$.
If propulsion is aligned with velocity, the equation is usually written as

$$m\left|\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right| = -(gI_{sp})\frac{\mathrm{d}m}{\mathrm{d}t}.$$

Time scale

Relative to the time scale of the orbit (where period is usually measured in hours), chemical propulsion usually happens in several seconds once in orbit, during which the gravitational acceleration may be ignored.

Dot-multiplying the equation below with $\vec{u}$

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{gI_{sp}}{m}\frac{\mathrm{d}m}{\mathrm{d}t}\vec{u},$$

we get

$$\vec{u} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{gI_{sp}}{m}\frac{\mathrm{d}m}{\mathrm{d}t}.$$

Integrating, we again obtain

$$\vec{u} \cdot (\vec{v}(t) - \vec{v}(0)) = gI_{sp} \ln \frac{m(0)}{m(t)}.$$

Small burns

For a sufficiently small burns, if $\vec{u}$ is in the same direction as $\vec{v}$,

$$|\underbrace{\vec{v}(t) - \vec{v}(0)}_{\Delta\vec{v}}| = gI_{sp} \ln \frac{m(0)}{m(t)}.$$

Then

$$\Delta v = |\Delta\vec{v}| = gI_{sp} \ln \frac{m(0)}{m(t)}$$

$$m(0) \exp\left(-\frac{\Delta v}{gI_{sp}}\right) = m(t).$$

$$m(0)\left[1 - \exp\left(-\frac{\Delta v}{gI_{sp}}\right)\right] = \underbrace{m(0) - m(t)}_{\Delta m}.$$

Suppose $\Delta v \ll gI_{sp}$,

$$\Delta v \approx gI_{sp} \frac{\Delta m}{m(0)}.$$