standard deviation in density would exceed one-thousandth of the mean density? Assume that the number of particles in a given volume \( V \) is a random variable with a Poisson distribution; that is,

\[
p_k = \frac{(\mu V)^k}{k!} e^{-\mu V}
\]

is the probability of finding exactly \( k \) particles in the volume \( V \), where there are \( \mu \) particles per unit volume on the average. For air at 68°F and one atmosphere pressure, \( \mu = 2.7 \times 10^{19} \) particles per cubic centimeter.

**Answer.** The sample would be a cube with side equal to \( 3.3 \times 10^{-5} \) cm.

### 10.6 Common probability density functions

**Uniform density function.** The simplest density function for a random scalar is the uniform distribution:

\[
p(x) = \begin{cases} 
\frac{1}{c}, & b - \frac{c}{2} \leq x \leq b + \frac{c}{2}, \\
0, & x > b + \frac{c}{2}, x < b - \frac{c}{2}.
\end{cases}
\]  

(10.6.1)

Obviously, we have

\[
\int_{-a}^{a} p(x) \, dx = 1,
\]  

(10.6.2)

\[
E(x) = b,
\]  

(10.6.3)

\[
E(x - b)^2 = c^2/12.
\]  

(10.6.4)

![Figure 10.6.1.](image)

**Gaussian density function for a random scalar.** Perhaps the most common distribution for a random scalar is the gaussian distribution:

\[
p(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x - \bar{x})^2}{2\sigma^2}}
\]  

for \( x \) in the range \( \bar{x} - 3\sigma \) to \( \bar{x} + 3\sigma \).

Tables of this form are found in many problem solution manuals.
would exceed one-thousandth the number of particles in a given distribution, that is,

$$V^k = k! \frac{e^{-\mu V}}{\mu^k}$$

exactly $k$ particles in the volume $V$ of unit volume on the average. Thus, its pressure, $\mu = 2.7 \times 10^{10}$ particles per cm$^3$.

be a cube with side equal to $2b$.

**Functions**

Simplest density function in general:

$$p(x) dx = 1,$$

$$x = b,$$

$$b^2 = \sigma^2/12.$$

$$-b + \frac{c}{2} \leq x \leq b + \frac{c}{2},$$

$$b + \frac{c}{2} < x < b - \frac{c}{2}.$$

$$E(x) = \bar{x},$$

$$E(x - \bar{x})^2 = \sigma^2.$$

**Figure 10.6.2.** The gaussian density function.

A convenient for representing many complicated phenomena by density functions lies in the central limit theorem,† which states that if $x$ is the sum of $N$ independent random quantities having density functions, then $x$ tends to have a gaussian density as $N \to \infty$ (see Problems 1 and 2). The probability that $x$ lies between $\bar{x} - \xi$ and $\bar{x} + \xi$ is given by

$$\int_{\bar{x} - \xi}^{\bar{x} + \xi} p(x) dx = \frac{1}{(2\pi)^{1/2} \sigma} \int_{-\xi}^{\xi} e^{-t^2/(2\sigma^2)} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\xi \sqrt{2}\sigma} e^{-u^2} du \leq \text{erf}(\xi/\sqrt{2}\sigma).$$


<table>
<thead>
<tr>
<th>$b$</th>
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<tbody>
<tr>
<td>$\xi/2$</td>
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<tr>
<td>$x$</td>
</tr>
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</table>

a random scalar. Perhaps the most well-known scalar is the gaussian distribution.
The "three sigma" \((3\sigma)\) value is often used in practical problems as virtually the upper bound on the variation from the mean, since the probability that \(x\) lies between \(-3\sigma\) and \(+3\sigma\) is .997. Analogous to the concept of a generating function for the mass function, a characteristic function for the density function of a random variable is defined by

\[
M_x(jv) \triangleq E(e^{jvx}) = \int_{-\infty}^{\infty} e^{jvx} p(x) \, dx, \quad j = \sqrt{-1},
\]

which is just the Fourier transform of the density function. It can be easily verified that

\[
E(x^n) = (-j)^n \frac{d^n M_x(jv)}{dv^n} \bigg|_{v=0}
\]

**Problem 1.** Using the results of Problem 1, Section 10.4, consider the case in which \(x_1\) and \(x_2\) are independent random scalars, each uniformly distributed on the interval \((-\frac{1}{2}, \frac{1}{2})\). With \(y = x_1 + x_2\), show that

\[
p(y) = \begin{cases} 
1 - |y|, & |y| < 1, \\
0, & |y| > 1.
\end{cases}
\]

![Figure 10.6.3. Density function of the sum of two uniformly distributed random variables.](image)

**Problem 2.** Using the results of Problem 1 (and Problem 1 of Section 10.4 again), consider the case in which \(x_1, x_2,\) and \(x_3\) are independent random scalars, each uniformly distributed on the interval \((-1, 1)\). With \(y = x_1 + x_2 + x_3\), show that
Concepts of Probability — Gaussian Density Function for a Random Vector

\[ p(y) = \begin{cases} \frac{3}{2} - y^2, & 0 \leq |y| \leq \frac{1}{4}, \\ \frac{3}{8} (\frac{3}{2} - |y|)^2, & \frac{1}{4} \leq |y| \leq \frac{3}{4}, \\ 0, & |y| > \frac{3}{4}. \end{cases} \]

Function 10.6.4. Density function of the sum of three uniformly distributed random variables.

As \( p(y) \) is tending toward a gaussian distribution, as indicated by the central limit theorem.

Show that the characteristic function of a gaussian random variable is

\[ M_x(jv) = \exp \left[ jv\bar{x} - \frac{v^2\sigma^2}{2} \right]. \]

**Gaussian density function for a random vector**

For a random \( n \)-vector, where the components can take on a continuous set of values, the most common probability density function, encountered in practice, and certainly the most important for this book, is the gaussian or normal distribution:

\[ p(x) = \frac{1}{(2\pi)^{n/2}|P|^{1/2}} \exp \left[ -\frac{1}{2} (x - \bar{x})^T P^{-1} (x - \bar{x}) \right]. \quad (10.7.1) \]

It can be shown that:

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x) \, dx_1 \cdots dx_n = 1, \quad (10.7.2) \]

\[ E(x) = \bar{x} = \text{mean value of vector,} \quad (10.7.3) \]

\[ E[x - \bar{x}] (x - \bar{x})^T = P = \text{covariance matrix of vector,} \quad (10.7.4) \]

The most common shorthand notation for this is "\( x \) is \( N(\bar{x}, P) \)."

where $|P|$ is the determinant of $P$, $P^{-1}$ is the matrix inverse of $P$. Note that $p(x)$ is completely characterized by giving only $\bar{x}$ and $P$.

If $P$ is a diagonal matrix, then $x - \bar{x}$ has components that are statistically independent, since $p(x)$ may then be factored into a product of $n$ scalar normal distributions. In other words, if the components of a Gaussian random vector are uncorrelated, they are statistically independent. By virtue of its definition, $P$ is a nonnegative definite matrix; i.e., it has positive (or zero) eigenvalues. Hence, by an orthogonal transformation, $S$,

$$ y = S(x - \bar{x}), $$

(10.7.5)

it is always possible to diagonalize $P$. Another way of saying this is that the hypersurfaces of constant likelihood (constant values of probability density) in the $x$-space are hyperellipsoids, and, by a rotation of axes, it is possible to use the principal axes of these hyperellipsoids as coordinate axes.

We are often interested in the probability that $x$ lies inside the hyperellipsoid:

$$ (x - \bar{x})^T P^{-1} (x - \bar{x}) = l^2, $$

(10.7.6)

where $l$ is a constant. By transforming to principal axes, this expression becomes

$$ \frac{y_1^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} + \cdots + \frac{y_n^2}{\sigma_n^2} = l^2. $$

(10.7.7)

By another transformation, $z_i = (y_i/\sigma_i)$, this expression becomes the equation for a hypersphere in $n$ dimensions:

$$ z_1^2 + z_2^2 + \cdots + z_n^2 = r^2. $$

(10.7.8)

The probability of finding $z$ inside this hypersphere is

$$ \int \int \cdots \int \frac{1}{(2\pi)^{n/2}} \exp \left\{ -\frac{1}{2} \left[ z_1^2 + \cdots + z_n^2 \right] \right\} dz_1, \ldots, dz_n, $$

(10.7.9)

where the integration is carried out over the volume $V$ of the hypersphere, $r$, where

$$ r^2 = z_1^2 + z_2^2 + \cdots + z_n^2. $$

(10.7.10)

In the $z$ space $|P| = 1$, since all the variances are unity and all covariances are zero. Thus the probability of finding $x$ inside the hyperellipsoid $(x - \bar{x})^T P^{-1} (x - \bar{x}) = l^2$ is

$$ \left[ \frac{1}{(2\pi)^{n/2}} \right] \int_0^l \exp \left( -\frac{1}{2} r^2 \right) f(r) \, dr, $$

(10.7.11)
Some Concepts of Probability — II

of $P$, $P^{-1}$ is the matrix inverse of $P$ characterized by giving unit volume when $x - \bar{x}$ has components that are all equal. \(x\) may then be factored into independent components. In other words, if the components of $x$ are uncorrelated, then $x - \bar{x}$ is Gaussian.

$S(x - \bar{x}) = \Sigma(x - \bar{x})$, normalize $P$. Another way of expressing the constant likelihood (constant volume) in the Gaussian is that the ellipsoids are spheres, and, by rotation, the principal axes of these hyperspheres

in the probability that $x$ lies inside the

$P^{-1}(x - \bar{x}) = l^2$.

Transforming to principal axes, the expression becomes

\[
\sum_{i=1}^{n} \frac{z_i^2}{\sigma_i^2} = l^2.
\]

\(z_i = (x_i - \bar{x}_i)\), this expression becomes

in $n$ dimensions:

\[
z_1^2 + \cdots + z_n^2 = r^2.
\]

inside this hypersphere is

\[
\int \cdots \int \frac{1}{\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2} r^2\right) \, dr,
\]

tried out over the volume $V$ of the space

\[
z_1^2 + z_2^2 + \cdots + z_n^2 = r^2,
\]

since all the variances are unit and above, the probability of finding $x$ inside the hypersphere $P$ is

\[
\int_0^l \exp\left(-\frac{1}{2} r^2\right) f(r) \, dr,
\]

\[
f(r) \, dr
\]

is the spherically symmetric volume element in an $n$-dimensional space. For $n = 1, 2, 3$, this probability is given by

\[
\begin{align*}
1: & \quad \sqrt{2/\pi} \int_0^l \exp\left(-\frac{1}{2} r^2\right) \, dr = \text{erf}(l/\sqrt{2}), \\
2: & \quad \int_0^l \exp\left(-\frac{1}{2} r^2\right) r \, dr = 1 - \exp\left(-\frac{1}{2} l^2\right), \\
3: & \quad \sqrt{2/\pi} \int_0^l \exp\left(-\frac{1}{2} r^2\right) r^2 \, dr = \frac{\text{erf}(l/\sqrt{2}) - \sqrt{2/\pi} l \exp\left(-\frac{1}{2} l^2\right)}{\sqrt{2/\pi}}.
\end{align*}
\]

Particular interest are the values for $l = 1, 2, 3$

<table>
<thead>
<tr>
<th>$n/l$</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tbody>
<tr>
<td>1</td>
<td>.683</td>
<td>.955</td>
<td>.997</td>
</tr>
<tr>
<td>2</td>
<td>.394</td>
<td>.865</td>
<td>.989</td>
</tr>
<tr>
<td>3</td>
<td>.200</td>
<td>.739</td>
<td>.971</td>
</tr>
</tbody>
</table>

These are often called the one-, two-, or three-sigma probabilities.

Consider a normally distributed two-dimensional vector with

and

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Eigenvectors of this covariance matrix are given by

\[
\begin{bmatrix} 4 - \sigma_1^2, 1 \\ 1, 1 - \sigma_1^2 \end{bmatrix} = 0
\]

\[
\sigma_1^2 - 5\sigma_1 + 3 = 0, \quad \Rightarrow \sigma_1^2 = 4.30, \quad \sigma_2^2 = .70,
\]

the eigenvectors are proportional to

\[
\begin{bmatrix} 1 \\ .30 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -3.30 \end{bmatrix}.
\]

Likelihood ellipses

\[
(x_1, x_2) \begin{bmatrix} 4, 1 \\ 1, 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = l^2
\]

shown in Figure 10.7.1 for $l = 1, 2, 3$. The probability of finding $x$ in the $l = 1$ ellipse is .394, inside the $l = 2$ ellipse is .865, and outside the $l = 3$ ellipse is .989.
An important property of gaussian random vectors. The remaining part of this book depends heavily on one important property of gaussian random vectors; that is, a linear combination of gaussian random vectors is also a gaussian random vector. Stated analytically, if \( x \) is a gaussian random vector with mean \( \bar{x} \) and covariance \( P_x \), and \( y = Ax + b \) where \( A \) is a constant matrix and \( b \) is a constant vector, then \( y \) is a gaussian random vector with mean \( \bar{y} \) and covariance \( P_y \), where

\[
\bar{y} = A\bar{x} + b, \tag{10.7.13}
\]

\[
P_y = AP_x A^T. \tag{10.7.14}
\]

The relations (10.7.13) and (10.7.14) follow very simply from the definition of expected values:

\[
\bar{y} = E(y) = \int_{-\infty}^{\infty} \cdots \int (Ax + b)p(x)\,dx_1 \cdots dx_n
\]

\[
= A\int_{-\infty}^{\infty} \cdots \int xp(x)\,dx_1 \cdots dx_n + b \int_{-\infty}^{\infty} \cdots \int p(x)\,dx_1 \cdots dx_n
\]

\[
= A\bar{x} + b,
\]

and

\[
P_y = E[(y - \bar{y})(y - \bar{y})^T] = A P_x A^T.
\]
Gaussian Density Function for a Random Vector

\[ p(y) = \left[ \pi \det(A^T A)^{-1} \right]^{n/2} \exp \left\{ -\frac{1}{2} (y - \hat{y})^T A^{-1} P_y^{-1} A^{-1} (y - \hat{y}) \right\} \]  \hspace{1cm} (10.7.17)

which was to be shown.

1. Define the joint characteristic function of a random vector using (10.6.10) via the use of multidimensional Fourier transforms and show that, for gaussian \( x, \)

\[ M_x(j\nu) = \exp \left\{ j\nu^T \hat{x} - \frac{1}{2} \nu^T \nu \right\} \]  \hspace{1cm} (Davenport and Root, *Introduction to Random Signals and Noise*, McGraw-Hill, 1958, p. 153.)

2. Prove (10.7.13) and (10.7.14) for arbitrary \( A \) by using the result of Problem 1 (see Cramer (1946), p. 312).

3. If \( b \) is a gaussian random vector independent of \( x, \) with mean \( \hat{b} \) and covariance \( P_b, \) show that Equations (10.7.13) and (10.7.14) are modified to

...