MAE 142 Air Vehicles. . .
Take-Home Final

Due date/time: Saturday, 14 Dec., 2024; 6pm

Please note that an additional two-hour window for late arrivals will be provided, but this extension deadline is a hard limit

You must show all work, including your codes, in order to get credit.

1. (15 points) Consider our aircraft from class and homework, with lift, drag, attitude and thrust such that it is making a constant-speed, constant-altitude turn. Suppose the average wind velocity is zero, and that the (constant) groundspeed is $\bar{s} = 0.26$ km/sec, and that the (constant) angular rate is $\omega = 0.026$ radians/sec (turning counterclockwise from when looking from above), yielding a turn radius of approximately $R = 10$ km. Suppose the (constant) altitude is $H = 5$ km. Let the nominal vehicle dynamics model and nominal/expected initial state be

$$\frac{dx}{dt}(t) = \frac{d}{dt}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}(t) = \hat{A}x(t) + \hat{B}u(t) \doteq \begin{pmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{pmatrix} x(t) + \begin{pmatrix} 0_3 \\ I_3 \end{pmatrix} u(t),$$
$$\bar{x}'(0) = (\bar{r}'(0), \bar{v}'(0)) = \left(R/2,\ 0,\ 0,\ 0,\ \bar{s},\ 0\right),$$

where $r = (r_1, r_2, r_3)'$ denotes the aircraft position, $v = (v_1, v_2, v_3)'$ denotes the aircraft velocity, $u = (u_1, u_2, u_3)'$ denotes the nominal aircraft acceleration (control), I_3 denotes the 3-by-3 identity matrix and 0_3 denotes the 3-by-3 zero matrix. In particular, $u(t) = -\omega^2 r(t)$ km/sec^2. Let us take a simple model for the noise process, where in particular, we let the actual dynamics be given by

$$\frac{d\hat{X}}{dt}(t) = \hat{A}\hat{X}(t) + \hat{B}u(t) + \begin{pmatrix} 0_3 \\ \hat{\sigma} I_3 \end{pmatrix} \dot{\hat{w}}(t),$$

where $\hat{\sigma} = 0.005$, and $\dot{\hat{w}}(t)$ formally denotes the "derivative" of a three-dimensional Brownian motion. Note that the expectation of $\hat{X}(0)$ is $\mathbb{E}[\hat{X}(0)] = \bar{x}(0)$.

Let the covariance of the initial state be

$$C_0 = \begin{pmatrix} c_{1,0} I_3 & c_{2,0} I_3 \\ c_{2,0} I_3 & c_{3,0} I_3 \end{pmatrix},$$

with $c_{1,0} = 0.001$ km^2, $c_{2,0} = 0$ and $c_{3,0} = 0.0001$ km^2/sec^2. (N.B.: this is even more unrealistic wrt the third and sixth components than the other details of the problem, but it's simpler for our needs here.)

We discretize time as $\{t_0, t_1, t_2 \ldots\}$ with $t_0 = 0$ and $t_{k+1} - t_k = \Delta_t \doteq 1.0$ sec for all $k \geq 0$. The discretized model of the true dynamics is

$$X_{k+1} = AX_k + \begin{pmatrix} 0_3 \\ \sigma I_3 \end{pmatrix} w_k, \quad X_0 \sim \mathcal{N}(\bar{x}(0), C_0), \tag{1}$$

where $\sigma \doteq \sqrt{\Delta_t}\,\hat{\sigma}$ and the w_k are independent, identically-distributed random variables, all with mean zero and covariance $C_w = I_3$.

(a) What should A be?

(b) Plot $r_2(t)$ versus $r_1(t)$ for $t \in [0, 2\pi/\omega]$. In this same plot window, using some markers, say points or circles, plot the zero-noise trajectory points over this same time interval. That is, plot the second versus the first components of what one obtains from $X_{k+1} = AX_k$ for $k = 0, 1, 2 \ldots K$ for appropriate K, with $X_0 = \bar{x}(0)$.

(c) In this same plot window, using different markers, over this same time interval, plot the second versus the first components of what one obtains from (1) for $k = 0, 1, 2 \ldots K$ for appropriate K, with $X_0 = \bar{x}(0)$ and the random sequence, w_k, being generated with the aid of the "randn" function in matlab.

(d) In a separate plot window, make 1000 runs, each with a new random sequence, and plot all the resulting first, second and third components of X_{25} and of X_{100} values as points and circles (or differing markers of your own choice), respectively. (That is, use matlab's plot3 function.) Do the same for the fourth, fifth and sixth components.

(e) Propagate the covariances of the X_k, which we will denote by C_k, by the method we obtained in class. Discuss plots you obtained in response to question 1d in the context of the standard deviations obtained here.

(f) Discuss the nonzero components of the off-diagonal blocks of the C_k covariance matrices. By "off-diagonal blocks", we mean the $[C_k]_{i,j}$ elements of the matrices for $i \in \{1, 2, 3\}$ with $j \in \{4, 5, 6\}$ and vice-versa.

2. (15 points) Now suppose we have some satellite-based observations. Consider the same situation as in problem 1, but with the following changes.

- Take $\hat{\sigma} = 0.005$ (still with $\sigma \doteq \sqrt{\Delta_t}\,\hat{\sigma}$).
- Let $c_{1,0} = 10.0$ km^2, $c_{2,0} = 0$ and $c_{3,0} = 1.0$ km^2/sec^2.
- Let $\Delta_t = 4.0$ sec (still with $C_w = I_3$ though).

For simplicity, we will assume that the satellites are in fixed positions, with states given by

$$x^{s,1} = \begin{pmatrix} 8000 \\ 100 \\ 800 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^{s,2} = \begin{pmatrix} 100 \\ 8000 \\ 800 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^{s,3} = \begin{pmatrix} 1000 \\ 2000 \\ 5000 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x^{s,4} = \begin{pmatrix} 2000 \\ 100 \\ 2000 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Suppose the satellites emit signals exactly once per time-step and all at the same moment for simplicity. At each time t_{k+1}, the air vehicle receives the four observations

$$y^j_{k+1} = |r^{k+1} - r^{s,j}| + t_{k+1} + \nu^j_{k+1},$$

where t_{k+1} denotes the signal emission time (i.e., $(k+1)\Delta_t$), r^{k+1} denotes the actual air vehicle position at time t_{k+1} (i.e., the first three components of its actual state, generated from equation (1)), $r^{s,j}$ denotes the position of the j^{th} satellite and ν^j_{k+1} denotes the noise affecting the y^j_{k+1} observation. Let the variance in the noise be $\sigma^2 = 10^{-4}$ for each observation. Note that we are taking the speed of light to be $c = 1.0$ km/sec for simplicity, and to avoid any concerns with catastrophic subtraction. At each time-step, you should take the a priori variance of t_{k+1} to be $\sigma_t^2 = 100.0$ sec, roughly corresponding to very little knowledge of the actual signal emission time. (With regard to this seemingly large value of $\sigma_t^2 = 100.0$, keep in mind that we're simplifying with $c = 1.0$ km/sec!)

(a) What should H_{k+1} be? Note that in the observation-update portion of the Kalman filter for navigation with this type of observation, you should append your state to include the time of signal emission. If you process each observation separately, this yields, in the case of our air vehicle model, a 1×7 H_{k+1}, three components each for position and velocity, and the one component for signal emission time. If you process all satellite observations together, H_{k+1} will have dimension $n_{sat} \times 7$, where n_{sat} is the number of satellites generating observations.

N.B.: The a priori covariance will also need to be appended with σ_t^2 in the $(7,7)$ entry and zeros in the $(j,7)$ and $(7,j)$ entires for $j \in \{1, 2 \dots 6\}$.) Further, in that case, you will need to keep the emission-time estimate as part of the state estimate between each of the observation updates, until all three or four of these have been completed. You should drop the emission-time component of the state estimate and covariance, before continuing with the dynamics update. That is, during the observation updates, you work with a seven-dimensional state, and during the dynamics updates, you work with a six-dimensional state.

(b) Suppose you only took one observation step, and no dynamics propagation, that is, you only take the satellite-based observations of the air vehicle initial state. In the case of observations by all four satellites, what would the resulting a posteriori covariance for the air vehicle state be? What would it be if you only had observations from the first three satellites?

(c) Simulate the system again, and plot the resulting observations, subtracting the time of each observation from them in order to normalize the plots a bit.

(d) The steps below are the final boss takedown.

(e) Build the Kalman-filter based navigator for this problem, and embed it in your simulation.

(f) Run your resulting navigator simulation with all four satellites. Use matlab's plot3 to generate the following, all in the same plot window. First plot the nominal/zero-noise air vehicle trajectory. In the same window, with markers such as red points, plot the position-components of the discrete-time model trajectory given by equation (1). In the same window, with different markers, plot the a priori position-components of the state trajectory estimate. In the same window, with yet different markers, plot the a posteriori position-components of the state trajectory estimate. Generate the same plots for the case where only the first three satellites are functioning.

(g) Create analogous plots as in item 2f, but for the velocity components instead of position. Was the satellite data useful for estimating the velocity?

(h) What were the a priori and a posteriori state-covariance matrices at the terminal time? In words compare the square-roots of the diagonal elements of these matrices with the errors in the state estimates at the terminal time.

(i) Could you propagate the a priori and a posteriori covariance matrices without actually generating the observations?