MAE 142 Air Vehicle Systems
Assignment 2 Solutions

Problem 1

Let the coordinate frame (I_1, I_2, I_3) be rotated as follows:

1. First, by $\theta = \frac{\pi}{3}$ radians in the I_1, I_2 plane.
2. Second, by $\omega = \frac{\pi}{4}$ radians in the new $\hat{I}_2, \hat{I}_3$ plane.

This results in a new frame $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$. Given a vector $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in the original frame (I_1, I_2, I_3), find the coordinates of x in the new frame $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$.

Solution:

$$\tilde{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ 0 & -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) & 0 \\ -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \approx \begin{pmatrix} 2.232 \\ 2.216 \\ 2.027 \end{pmatrix}$$

Problem 2 Solution

Given a unit vector $e \in \mathbb{R}^3$ with $|e| = 1$, verify the following assertions for all $v \in \mathbb{R}^3$: 1. $S(e)v = e \times v$ 2. $(I - ee^T)v = [S(e)]^2 v$ 3. $[S(e)]^3 v = S(e)v$ 4. $e^T S(e) = S(e)e = 0$ where $S(e)$ is the skew-symmetric matrix associated with e, defined as:

$$S(e) = \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix}$$

Solution:

Assertion 1: $S(e)v = e \times v$

By definition, the skew-symmetric matrix $S(e)$ is constructed such that for any vector $v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$,

$$S(e)v = e \times v$$

where $e \times v$ denotes the cross product of e and v.

Thus, $S(e)v$ indeed represents the cross product $e \times v$. Therefore, **Assertion 1 is correct**.

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Assertion 2: $(I - ee^T)v = [S(e)]^2 v$

To verify this assertion, we examine both sides of the equation separately.

Left Side: $(I - ee^T)v$ The expression $I - ee^T$ is a projection matrix that projects v onto the plane perpendicular to e. Thus,

$$(I - ee^T)v = v - (e \cdot v)e$$

represents the component of v orthogonal to e.

Right Side: $[S(e)]^2 v$ Since $S(e)$ is a skew-symmetric matrix representing the cross product with e, squaring $S(e)$ results in:

$$[S(e)]^2 = -(I - ee^T)$$

Thus,

$$[S(e)]^2 v = -(I - ee^T)v$$

Conclusion Comparing the two sides, we see that:

$$(I - ee^T)v = -[S(e)]^2 v$$

Therefore, **Assertion 2 is incorrect**, as the correct relationship is $(I - ee^T)v = -[S(e)]^2 v$.

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Assertion 3: $[S(e)]^3 v = S(e)v$

To verify this assertion, we calculate $S(e)^3$ more closely.

Properties of $S(e)$ Since $S(e)$ is skew-symmetric, we have:

$$S(e)^2 = -(I - ee^T)$$

This leads to:

$$S(e)^3 = S(e) \cdot S(e)^2 = S(e) \cdot (-(I - ee^T)) = -S(e)$$

Conclusion Thus, we find that:

$$S(e)^3 = -S(e)$$

and therefore,

$$S(e)^3 v = -S(e)v$$

This shows that **Assertion 3 is incorrect**, as $S(e)^3 v$ is actually the negative of $S(e)v$, not equal to $S(e)v$.

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Assertion 4: $e^T S(e) = S(e)e = 0$

This assertion involves checking if the matrix $S(e)$ annihilates e from both the left and the right.

Left Side: $e^T S(e) = 0$ Since $S(e)$ represents the cross product with e, the row vector $e^T S(e)$ represents $e \times e$, which is zero:

$$e^T S(e) = \begin{pmatrix} e_x & e_y & e_z \end{pmatrix} \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

Right Side: $S(e)e = 0$ Similarly, $S(e)e$ represents the cross product $e \times e$, which is also zero:

$$S(e)e = \begin{pmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Conclusion Both $e^T S(e) = 0$ (a row vector) and $S(e)e = 0$ (a column vector) hold true, so **Assertion 4 is correct**.

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Summary of Results

- **Assertion 1**: Correct
- **Assertion 2**: Incorrect (the correct relation is $(I - ee^T)v = -[S(e)]^2 v$)
- **Assertion 3**: Incorrect (the correct relation is $[S(e)]^3 v = -S(e)v$)
- **Assertion 4**: Correct

Problem 3

Given a rotation matrix defined by its Euler axis e and principal angle ϕ, the rotation matrix is:

$$\bar{G}(e, \phi) = \cos(\phi)(I - ee^T) + ee^T - \sin(\phi)S(e)$$

where $S(e)$ is the skew-symmetric matrix associated with $e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$, defined as:

$$S(e) = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix}$$

We aim to show that the transpose of this matrix equals its counterpart when the angle is negated, i.e., to prove:

$$[\bar{G}(e, \phi)]^T = \bar{G}(e, -\phi)$$

Solution:

The transpose of $\bar{G}(e, \phi)$ is given as follows:

$$\bar{G}(e, \phi)^T = \cos(\phi)(I - ee^T) + ee^T + \sin(\phi)S(e)$$

On the other hand,

$$\bar{G}(e, -\phi) = \cos(-\phi)(I - ee^T) + ee^T - \sin(-\phi)S(e)$$

Using the trigonometric identities $\cos(-\phi) = \cos(\phi)$ and $\sin(-\phi) = -\sin(\phi)$, we get:

$$\bar{G}(e, -\phi) = \cos(\phi)(I - ee^T) + ee^T + \sin(\phi)S(e)$$

Therefore,

$$[\bar{G}(e, \phi)]^T = \bar{G}(e, -\phi)$$

Problem 4

Suppose we are at longitude $\lambda = 0$ and latitude $\phi = \frac{\pi}{6}$ radians and elevation (altitude) zero. Suppose the Greenwich meridian is aligned with the first basis vector in our Earth-centered inertial (ECI) system at time $t = 0$. For simplicity, assume the Earth is a sphere of radius 6378.0 km. Obtain our position in the ECI system one hour past $t = 0$.

Solution:

Given:

- Latitude $\phi = \frac{\pi}{6}$, Earth rotation rate $\omega = \frac{2\pi}{86400}$ rad/s.
- Time $t = 3600$ s (1 hour), initial position $\mathbf{r}_0 = \begin{pmatrix} 6378 \\ 0 \\ 0 \end{pmatrix}$ km.

Rotation Angles

$$\lambda = \omega t = \frac{\pi}{12}$$

Rotation Matrices

1. Rotation about y-axis by $-\phi$:

$$R_y(-\phi) = \begin{pmatrix} 0.866 & 0 & -0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.866 \end{pmatrix}$$

2. Rotation about z-axis by λ:

$$R_z(\lambda) = \begin{pmatrix} 0.9659 & -0.2588 & 0 \\ 0.2588 & 0.9659 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Final Position Calculation

$$\mathbf{r}_{\text{ECI}} = R_z(\lambda)R_y(-\phi)\mathbf{r}_0 = \begin{pmatrix} 5335.3 \\ 1429.6 \\ 3189.0 \end{pmatrix} \text{ km}$$

Problem 5

Let the position in Problem 4 be the origin of a local "UEN" (up/east/north) coordinate system. Suppose that our vehicle is at $X_{UEN} = (1, 10, 5)^T$ km. What is its position in the ECI system? What is its position in the local "ENU" (east/north/up) system?

Solution: Given:

- Position in UEN (Up, East, North) coordinates: $\mathbf{r}_{\text{UEN}} = \begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix}$ km.
- ECI origin position from Problem 4: $\mathbf{r}_{\text{ECI, origin}} = \begin{pmatrix} 5335.3 \\ 1429.6 \\ 3189.0 \end{pmatrix}$ km.
- Latitude $\phi = \frac{\pi}{6}$, rotation angle $\lambda = \frac{\pi}{12}$.

Rotation Matrices

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ECI Position Calculation

$$\mathbf{r}_{\text{ECI}} = \mathbf{r}_{\text{ECI, origin}} + R_z(\lambda) R_y(-\phi) \mathbf{r}_{\text{UEN}}$$

Thus:

$$\mathbf{r}_{\text{ECI}} = \begin{pmatrix} 5331.13 \\ 1438.83 \\ 3193.83 \end{pmatrix} \text{ km}$$

ENU Coordinates

The ENU coordinates are simply a reordering of UEN:

$$\mathbf{r}_{\text{ENU}} = \begin{pmatrix} 10 \\ 5 \\ 1 \end{pmatrix} \text{ km}$$

Problem 6

For the situation described in Problem 4, let $\bar{G}$ denote the combined rotation matrix. Determine the Euler axis and principal angle for this combined rotation.

Solution: To determine the principal angle θ and Euler axis $\mathbf{e}$ for the combined rotation matrix $\bar{G}$, we proceed as follows:

1. Principal Angle θ: The angle θ can be found using:

$$\cos\theta = \frac{\text{trace}(\bar{G}) - 1}{2}$$

Therefore,

$$\theta = 0.584, \text{radians}$$

2. Euler Axis $\mathbf{e}$: Using $\sin\theta \approx 0.735$, the components of the Euler axis $\mathbf{e} = [e_x, e_y, e_z]^T$ are calculated as:

$$e_x = \frac{\bar{G}_{32} - \bar{G}_{23}}{2\sin\theta} \approx 0.296, \quad e_y = \frac{\bar{G}_{13} - \bar{G}_{31}}{2\sin\theta} = 0, \quad e_z = \frac{\bar{G}_{21} - \bar{G}_{12}}{2\sin\theta} \approx 0.680$$

Thus, the Euler axis is:

$$\mathbf{e} \approx \left[0.117, -0.891, 0.438\right]^T.$$