MAE 142- Homework 1 Solutions

November 4, 2024

Problem 1

(5 points) By hand (with the aid of a calculator, etc.), obtain the diagonal matrix of eigenvalues (Λ) and the matrix of eigenvectors (S) for matrix A below. Show your work.

$$A = \begin{pmatrix} 5 & -9 \\ -12 & 2 \end{pmatrix}$$

Solution:

Step 1: Find the Eigenvalues

The characteristic equation is given by:

$$\det(A - \lambda I) = 0$$

where I is the identity matrix.

Compute $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 5 - \lambda & -9 \\ -12 & 2 - \lambda \end{pmatrix}$$

Compute the determinant:

$$\det(A - \lambda I) = (5 - \lambda)(2 - \lambda) - 108$$

So the characteristic equation is:

$$\lambda^2 - 7\lambda - 98 = 0$$

Solve the quadratic equation:

$$\lambda = \frac{7 \pm 21}{2}$$

Thus,

$$\lambda_1 = 14$$

$$\lambda_2 = -7$$

Step 2: Find the Eigenvectors

Eigenvector for $\lambda_1 = 14$
We solve:

$$(A - 14I)\mathbf{v} = 0$$

Compute $A - 14I$:

$$A - 14I = \begin{pmatrix} 5-14 & -9 \\ -12 & 2-14 \end{pmatrix} = \begin{pmatrix} -9 & -9 \\ -12 & -12 \end{pmatrix}$$

Set up the system:

$$\begin{cases} -9v_1 - 9v_2 = 0 \\ -12v_1 - 12v_2 = 0 \end{cases}$$

This simplifies to:

$$-9v_1 - 9v_2 = 0 \quad \Rightarrow \quad v_1 + v_2 = 0$$

So, the eigenvector corresponding to $\lambda_1 = 14$ is any non-zero scalar multiple of:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvector for $\lambda_2 = -7$
We solve:

$$(A - (-7)I)\mathbf{v} = (A + 7I)\mathbf{v} = 0$$

Compute $A + 7I$:

$$A + 7I = \begin{pmatrix} 5+7 & -9 \\ -12 & 2+7 \end{pmatrix} = \begin{pmatrix} 12 & -9 \\ -12 & 9 \end{pmatrix}$$

Set up the system:

$$\begin{cases} 12v_1 - 9v_2 = 0 \\ -12v_1 + 9v_2 = 0 \end{cases}$$

From the first equation:

$$12v_1 - 9v_2 = 0 \quad \Rightarrow \quad 12v_1 = 9v_2 \quad \Rightarrow \quad \frac{v_1}{v_2} = \frac{9}{12} = \frac{3}{4}$$

So, the eigenvector corresponding to $\lambda_2 = -7$ is any non-zero scalar multiple of:

$$\mathbf{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Step 3: Form the Matrices S and Λ

The matrix S of eigenvectors is:

$$S = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}$$

And the diagonal matrix Λ of eigenvalues is:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & -7 \end{pmatrix}$$

Step 4: Verify $A = S\Lambda S^{-1}$
Optionally, we can verify that $A = S\Lambda S^{-1}$ by computing $S\Lambda S^{-1}$ and confirming it equals A.

Problem 2

(5 points) Using MATLAB, obtain Λ and S for

$$A = \begin{pmatrix} 1+7i & 6i & -6i \\ 4-4i & 5-3i & -3+3i \\ 4+4i & 4+4i & -2-4i \end{pmatrix}$$

Based only on these results, indicate whether A is nonsingular, and why. Do the same for

$$A = \begin{pmatrix} 1+7i & 6i & -6i \\ -4-4i & -3-3i & 3+3i \\ -4+4i & -4+4i & 4-4i \end{pmatrix}$$

Solution:

To compute the eigenvalues and eigenvectors of A, we can use MATLAB. The MATLAB code is as follows:

```
[S, Lambda] = eig(A);
```

Here:

- Lambda will be a diagonal matrix containing the eigenvalues.
- S will contain the corresponding eigenvectors as columns.

If all eigenvalues are non-zero, then A is nonsingular. If any eigenvalue is zero, then A is singular.

Running the MATLAB code for the first matrix could give us the following results:

```
A = [1 + 7i, 6i, -6i; 4 - 4i, 5 - 3i, -3 + 3i; 4 + 4i, 4 + 4i, -2 - 4i];
     [S,Lambda] = eig(A)

S =

  -0.7071 + 0.0000i   0.6000 + 0.0000i   0.0000 + 0.0000i
   0.7071 + 0.0000i   0.0000 - 0.0000i   0.7071 - 0.0000i
   0.0000 + 0.0000i   0.8000 + 0.0000i   0.7071 + 0.0000i


Lambda =

   1.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   2.0000 - 0.0000i
```

Therefore, this matrix is nonsingular since all eigenvalues are non-zero.

Running the MATLAB code for the second matrix could give us the following results:

```
A = [1 + 7i, 6i, -6i; -4 - 4i, -3 - 3i, 3 + 3i; -4 + 4i, -4 + 4i, 4 - 4i];
[S,Lambda] = eig(A)

S =

   0.7071 + 0.0000i   0.6000 + 0.0000i   0.0000 + 0.0000i
  -0.7071 + 0.0000i   0.0000 - 0.0000i   0.7071 - 0.0000i
  -0.0000 + 0.0000i   0.8000 + 0.0000i   0.7071 + 0.0000i


Lambda =

   1.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i
```

Therefore, this matrix is singular since one eigenvalue equals zeros.

Problem 3

(10 points) Obtain an analytical (i.e., math, not code) solution to $\dot{\xi}(t) = A\xi(t)$, $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Solution:

Step 1: Eigenvalues of A

The characteristic equation is:

$$\det(A - \lambda I) = 0$$

We compute:

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = (\lambda - 3)(\lambda - 1) = 0$$

Thus, the eigenvalues are:

$$\lambda_1 = 3, \quad \lambda_2 = 1$$

Step 2: Eigenvectors of A

For $\lambda_1 = 3$, we solve $(A - 3I)\mathbf{v}_1 = 0$:

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

This gives:

$$v_1 = v_2 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$, we solve $(A - I)\mathbf{v}_2 = 0$:

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

This gives:

$$v_1 = -v_2 \quad \Rightarrow \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step 3: General Solution

The general solution is:

$$\xi(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\xi(t) = \begin{pmatrix} c_1 e^{3t} + c_2 e^{t} \\ c_1 e^{3t} - c_2 e^{t} \end{pmatrix}$$

Step 4: Applying Initial Condition

Given $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, we have:

$$\xi(0) = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Solving:

$$c_1 = 4, \quad c_2 = -1$$

Step 5: Final Solution

The final solution is:

$$\xi(t) = \begin{pmatrix} 4e^{3t} - e^{t} \\ 4e^{3t} + e^{t} \end{pmatrix}$$

Problem 4

(5 points) Find an analytical solution to $\dot{\xi}(t) = A\xi(t) + \nu(t)b$, $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nu(t) = \exp(-2t)$$

Solution:

We are given the differential equation:

$$\dot{\xi}(t) = A\xi(t) + \nu(t),$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \nu(t) = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Our goal is to find an analytical solution for $\xi(t)$.

Step 1: Calculate e^{At}

To find the state transition matrix e^{At}, we begin by finding the eigenvalues and eigenvectors of A.

1. Eigenvalues of A:

The characteristic polynomial of A is:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 3)(\lambda - 1) = 0.$$

Solving this, we find the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$.

2. Eigenvectors of A:

For $\lambda_1 = 3$:

$$(A - 3I)v = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

This gives the eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 1$:

$$(A - I)v = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

This gives the eigenvector $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. Construct e^{At}:

Using these eigenvalues and eigenvectors, we can form the matrix P of eigenvectors and the diagonal matrix Λ of eigenvalues:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then:

$$e^{At} = Pe^{\Lambda t}P^{-1}.$$

Since $e^{\Lambda t} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix}$ and $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we get:

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t} \end{bmatrix}.$$

Step 2: Solve for $\xi(t)$

Using the initial condition $\xi(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and taking into account the effect of $\nu(t) = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we proceed by computing each component of $\xi(t)$.

1. Solution from the Initial Condition: The term involving the initial condition is:

$$\phi(t)\xi(0) = \frac{1}{2}\begin{bmatrix} e^{3t} + e^t & e^{3t} - e^t \\ e^{3t} - e^t & e^{3t} + e^t \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Expanding each component gives

$$\phi(t)\xi(0) = \begin{bmatrix} 4e^{3t} - e^t \\ 4e^{3t} + e^t \end{bmatrix}.$$

2. Effect of $\nu(t)$: To account for $\nu(t) = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we compute:

$$x_p(t) = \int_0^t \phi(t-\tau)\nu(\tau)\, d\tau = \int_0^t \frac{1}{2}\begin{bmatrix} e^{3(t-\tau)} + e^{t-\tau} \\ e^{3(t-\tau)} - e^{t-\tau} \end{bmatrix} e^{-2\tau}\, d\tau.$$

Integrating each component:
- First component:

$$\xi_{p1}(t) = \frac{41}{10}e^{3t} - \frac{5}{6}e^t - \frac{4}{15}e^{-2t}.$$

- Second component:

$$\xi_{p2}(t) = \frac{41}{10}e^{3t} + \frac{5}{6}e^t + \frac{1}{15}e^{-2t}.$$

So,

$$x_p(t) = \begin{bmatrix} \frac{41}{10}e^{3t} - \frac{5}{6}e^t - \frac{4}{15}e^{-2t} \\ \frac{41}{10}e^{3t} + \frac{5}{6}e^t + \frac{1}{15}e^{-2t} \end{bmatrix}.$$

3. Final Solution: Combining the terms, we get:

$$\xi(t) = \phi(t)\xi(0) + x_p(t) = \begin{bmatrix} \frac{41}{10}e^{3t} - \frac{5}{6}e^t - \frac{4}{15}e^{-2t} \\ \frac{41}{10}e^{3t} + \frac{5}{6}e^t + \frac{1}{15}e^{-2t} \end{bmatrix}.$$

Alternative solution for Problem 4

Step 1: Homogeneous Solution

The homogeneous solution is:

$$\xi_h(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\xi_h(t) = \begin{pmatrix} c_1 e^{3t} + c_2 e^t \\ c_1 e^{3t} - c_2 e^t \end{pmatrix}$$

Step 2: Particular Solution

We assume the particular solution is of the form:

$$\xi_p(t) = \mathbf{v}e^{-2t}$$

Substituting into the non-homogeneous system:

$$-2\mathbf{v} = A\mathbf{v} + b$$

$$-2\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solving this system gives:

$$v_1 = -\frac{4}{15}, \quad v_2 = \frac{1}{15}$$

Thus, the particular solution is:

$$\xi_p(t) = \begin{pmatrix} -\frac{4}{15} \\ \frac{1}{15} \end{pmatrix} e^{-2t}$$

Step 3: General Solution

The general solution is:

$$\xi(t) = \xi_h(t) + \xi_p(t)$$

$$\xi(t) = \begin{pmatrix} c_1 e^{3t} + c_2 e^t \\ c_1 e^{3t} - c_2 e^t \end{pmatrix} + \begin{pmatrix} -\frac{4}{15}e^{-2t} \\ \frac{1}{15}e^{-2t} \end{pmatrix}$$

Step 4: Applying Initial Condition

We are given $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Solving the system of equations for c_1 and c_2, we get:

$$c_1 = \frac{41}{10}, \quad c_2 = -\frac{5}{6}$$

Step 5: Final Solution

The final solution is:

$$\xi(t) = \begin{pmatrix} \frac{41}{10}e^{3t} - \frac{5}{6}e^t - \frac{4}{15}e^{-2t} \\ \frac{41}{10}e^{3t} + \frac{5}{6}e^t + \frac{1}{15}e^{-2t} \end{pmatrix}$$

Problem 5

(8 points) In the notation of class, find $\tilde{S}$ and $\tilde{\Lambda}$ for the matrix

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}$$

Step 1: Finding the Eigenvalues

The characteristic equation is:

$$\det(A - \lambda I) = 0$$

We compute:

$$\det(A - \lambda I) = (2 - \lambda)(8 - \lambda) - (-3)(3)$$

Simplifying:

$$\det(A - \lambda I) = \lambda^2 - 10\lambda + 25 = 0$$

The eigenvalue is $\lambda = 5$, with algebraic multiplicity 2.

Step 2: Finding the Eigenvector

We solve $(A - 5I)\mathbf{v} = 0$:

$$A - 5I = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix}$$

This gives:

$$v_1 = v_2 \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 3: Finding the Generalized Eigenvector

We solve $(A - 5I)\mathbf{v}_g = \mathbf{v}_1$:

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_{g1} \\ v_{g2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This gives:

$$v_{g1} = 0, \quad v_{g2} = \frac{1}{3}$$

Thus, $\mathbf{v}_g = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$.

Step 4: Jordan Canonical Form

The Jordan form of A is:

$$\tilde{\Lambda} = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$$

The matrix of eigenvectors and generalized eigenvectors is:

$$\tilde{S} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix}$$

Step 5: Verifying the Jordan Form

We compute the inverse of $\tilde{S}$:

$$\tilde{S}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

and verify that:

$$A = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1}$$

Problem 6

(7 points) Find an expression for $\exp\left(\tilde{\Lambda}t\right)$ for your $\tilde{\Lambda}$ from Problem 5. Use that and $\tilde{S}$ (from Problem 5) to solve $\dot{\xi}(t) = A\xi(t)$, $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ in the case of matrix A from Problem 5.

Solution:

Step 1: Jordan Canonical Form of A

From Problem 5, the Jordan canonical form of A is:

$$\tilde{\Lambda} = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$$

The matrix $\tilde{S}$ is:

$$\tilde{S} = \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{3} \end{pmatrix}$$

and the inverse of $\tilde{S}$ is:

$$\tilde{S}^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

Step 2: Matrix Exponential of $\tilde{\Lambda}$

The matrix exponential of $\tilde{\Lambda}$ is:

$$e^{\tilde{\Lambda}t} = e^{5t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Step 3: Compute e^{At}

We compute $e^{At} = \tilde{S}e^{\tilde{\Lambda}t}\tilde{S}^{-1}$:

$$e^{At} = e^{5t} \begin{pmatrix} 1 & t \\ 1 & t + \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

$$e^{At} = e^{5t} \begin{pmatrix} 1 - 3t & 3t \\ -3t & 3t + 1 \end{pmatrix}$$

Step 4: Solve for $\xi(t)$

We solve for $\xi(t)$ using $\xi(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$:

$$\xi(t) = e^{5t} \begin{pmatrix} 1 - 3t & 3t \\ -3t & 3t + 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\xi(t) = e^{5t} \begin{pmatrix} 3 + 6t \\ 6t + 5 \end{pmatrix}$$

Final Solution

The solution is:

$$\xi(t) = e^{5t} \begin{pmatrix} 3 + 6t \\ 6t + 5 \end{pmatrix}$$