Problem 1.

Solution.

In an $n \times n$ linear system, to zero out an element in the first column, we need 1 division, n multiplication and n additions. To zero each of the bottom $n-1$ elements, we will thus need $(n-1)((n+1)\mathrm{M} + n\mathrm{A})$.

For the given 4×4 system, we will need

$$3(5\mathrm{M} + 4\mathrm{A}) + 2(4\mathrm{M} + 3\mathrm{A}) + 1(3\mathrm{M} + 2\mathrm{A}) = 26\mathrm{M} + 20\mathrm{A} = 46\mathrm{FLOPs}$$

for triangularization.

For back substitution, we need

$$1\mathrm{M} + (2\mathrm{M} + 1\mathrm{A}) + (3\mathrm{M} + 2\mathrm{A}) + (4\mathrm{M} + 3\mathrm{A}) = 10\mathrm{M} + 6\mathrm{A} = 16\mathrm{FLOPS}.$$

Hence 62 FLOPs is required to solve the given system.

Problem 2.

Solution.

For tridiagonal systems only one element need to be processed in each column, which needs 1 division, 2 multiplications and 2 additions.

For the given 6×6 tridiagonal system, we will need $(6-1)(3\mathrm{M} + 2\mathrm{A}) = 15\mathrm{M} + 10\mathrm{A} = 25\mathrm{FLOPs}$ for triangularization.

For back substitution, we need

$$1\mathrm{M} + (6-1)(2\mathrm{M} + 1\mathrm{A}) = 11\mathrm{M} + 5\mathrm{A} = 16\mathrm{FLOPS}.$$

Hence 41 FLOPs is required to solve the given system.

Problem 3.

Solution. Subdivide the interval $[0, 2]$ using step size $h = \frac{1}{3}$ and let $x_k = 0 + kh = kh$. Thus, $n = 6$. Denote $u(x_k) = u_k$ for $k = 0, \cdots, n$.

Using

$$u_{xx}(x_k) \approx \frac{u_{k-1} - 2u_k + u_{k+1}}{h^2},$$

we have

$$-\frac{1}{2}\frac{u_{k-1} - 2u_k + u_{k+1}}{h^2} + u_k = 1 + \sin(\tfrac{\pi x_k}{4}).$$

Rearranging, we have

$$-\frac{1}{2h^2}u_{k-1} + (1 + \frac{1}{h^2})u_k - \frac{1}{2h^2}u_{k+1} = 1 + \sin(\tfrac{\pi x_k}{4}).$$

Plugging in $h = \frac{1}{3}$, we have

$$-\frac{9}{2}u_{k-1} + 10u_k - \frac{9}{2}u_{k+1} = 1 + \sin(\tfrac{k\pi}{12}).$$

The boundary conditions are specified as $u_0 = 2$ and $u_2 = 42$. Therefore, for $k = 1$ and $k = n - 1 = 5$

$$\begin{aligned} 10u_1 - \frac{9}{2}u_2 &= 1 + \sin(\tfrac{\pi}{12}) + \frac{9}{2} \cdot 2 \\ -\frac{9}{2}u_4 + 10u_5 &= 1 + \sin(\tfrac{5\pi}{12}) + \frac{9}{2} \cdot 42. \end{aligned}$$

Rearranging the equation $k = 1, \cdots, 5$ into matrix form, we get

$$\begin{pmatrix} 10 & -\frac{9}{2} & & & \\ -\frac{9}{2} & 10 & -\frac{9}{2} & & \\ & -\frac{9}{2} & 10 & -\frac{9}{2} & \\ & & -\frac{9}{2} & 10 & -\frac{9}{2} \\ & & & -\frac{9}{2} & 10 & -\frac{9}{2} \\ & & & & -\frac{9}{2} & 10 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{pmatrix} 1 + \sin(\frac{\pi}{12}) + \frac{9}{2} \cdot 2 \\ 1 + \sin(\frac{2\pi}{12}) \\ 1 + \sin(\frac{3\pi}{12}) \\ 1 + \sin(\frac{4\pi}{12}) \\ 1 + \sin(\frac{5\pi}{12}) + \frac{9}{2} \cdot 42 \end{pmatrix} = \begin{pmatrix} 10 + \sin(\frac{\pi}{12}) \\ 1 + \sin(\frac{2\pi}{12}) \\ 1 + \sin(\frac{3\pi}{12}) \\ 1 + \sin(\frac{4\pi}{12}) \\ 190 + \sin(\frac{5\pi}{12}) \end{pmatrix}.$$

Problem 4.

Solution. As in Problem 3, we have

$$-\frac{1}{2h^2}u_{k-1} + (1+\frac{1}{h^2})u_k - \frac{1}{2h^2}u_{k+1} = 1+\sin(\tfrac{\pi x_k}{4}),$$

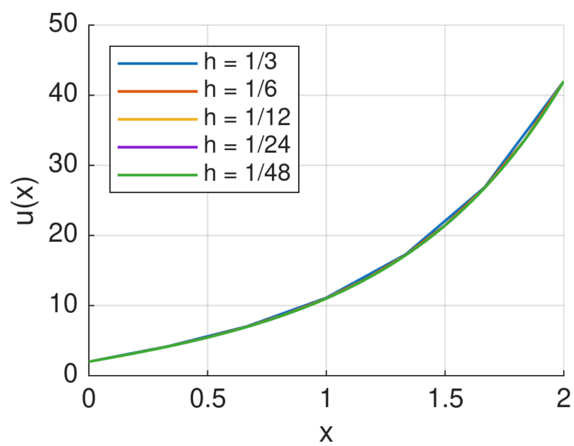
and for $k = 1$ and $n-1$, we have

$$(1+\frac{1}{h^2})u_1 - \frac{1}{2h^2}u_2 = 1+\sin(\tfrac{\pi}{12}) + \frac{1}{2h^2}\cdot u(0)$$
$$-\frac{1}{2h^2}u_4 + (1+\frac{1}{h^2})u_5 = 1+\sin(\tfrac{5\pi}{12}) + \frac{1}{2h^2}\cdot u(2).$$

Rearranging the equation $k = 1, \cdots, n-1$ into matrix form, we get

$$\begin{pmatrix} 1+\frac{1}{h^2} & -\frac{1}{2h^2} & & & & \\ -\frac{1}{2h^2} & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -\frac{1}{2h^2} \\ & & & & -\frac{1}{2h^2} & 1+\frac{1}{h^2} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 1+\sin(\frac{\pi}{4}h) + \frac{1}{2h^2}\cdot u(0) \\ 1+\sin(\frac{2\pi}{4}h) \\ \vdots \\ 1+\sin(\frac{(n-2)\pi}{12}) \\ 1+\sin(\frac{(n-1)\pi}{12}) + \frac{1}{2h^2}\cdot u(2) \end{pmatrix}.$$

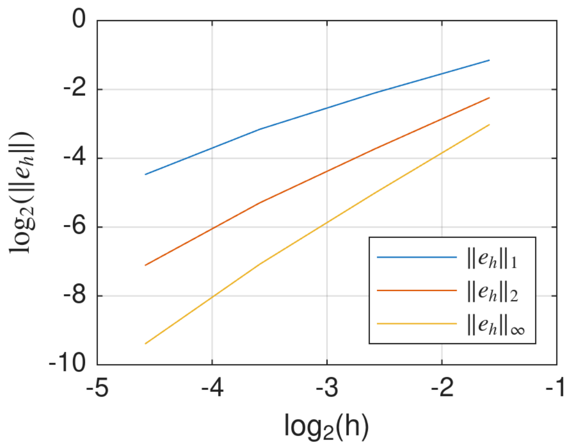
Solving this system in MATLAB for the given hs, we get



Using the most accurate solution (the one obtained using $h = 1/48$), the actual errors for the given hs are

h	$\|e_h\|_1$	$\|e_h\|_2$	$\|e_h\|_\infty$
1/3	0.45098	0.21193	0.12348
1/6	0.23284	0.07540	0.03100
1/12	0.11210	0.02549	0.00741
1/24	0.04497	0.00722	0.00148
$\mathcal{O}(h^?)$	1	1.5	2

Graphically, this is



and the order can be estimated from the slope in the log-log plot.

The order of approximation for u_{xx} is $\mathcal{O}(h^2)$. The error $\|e_h\|_\infty$ approaches 0 with the same order as u_{xx}. The order of $\|e_h\|_1$ and $\|e_h\|_2$ converges slower (has lower orders) because $\|e_h\|_1 \lesssim n\|e_h\|_\infty$ and $\|e_h\|_2 \lesssim \sqrt{n}\|e_h\|_\infty$.

The true solution is $\bar{u}(x) =$

$$\frac{1}{32+\pi^2} \cdot \left[\frac{1+e^{-\sqrt{2}t}}{-1+e^{4\sqrt{2}}}((32+\pi^2)(e^{4\sqrt{2}} - e^{\sqrt{2}t}) + (e^{\sqrt{2}(2+t)} - e^{2\sqrt{2}})(1280+41\pi^2)) + 32\sin(\frac{\pi t}{4}) \right].$$

Hence the actual error $e_h = u_h - [\bar{u}(x_1), \cdots, \bar{u}_{x_{n-1}}]$ for each h is as follows

h	$\|e_h\|_1$	$\|e_h\|_2$	$\|e_h\|_\infty$
1/3	0.45279	0.21277	0.12398
1/6	0.23656	0.07660	0.03150
1/12	0.11959	0.02720	0.00791
1/24	0.05996	0.00962	0.00198
1/48	0.03000	0.00340	0.00050
$\mathcal{O}(h^?)$	1	1.5	2