Problem 1.

Solution. We have $f(x_0) = f(1) = e$, $f(x_1) = f(2) = 2e^2$, and

$$\ell(x) = \frac{x_1 - x}{x_1 - x_0} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (2-x)e + 2(x-1)e^2 = \boxed{(2e^2 - e)x + 2e - 2e^2}$$
$$\overset{\text{OR}}{=} (2e^2 - e)(x-1) + e.$$

Since $f''(x) = (2+x)e^x$, the error bound is given by

$$e^{\text{la}} \le \frac{\max_{x\in[1,2]}|f''(x)|}{8}(x_1 - x_0)^2 = \frac{f''(2)}{8} = \boxed{\frac{1}{2}e^2 \approx 3.6945}.$$

The actual errors are

$$|f(1.25) - \ell(1.25)| \approx |4.3629 - 5.7332| = 1.3703,$$
$$|f(1.5) - \ell(1.5)| \approx |6.7225 - 8.7482| = 2.0257,$$
$$|f(1.75) - \ell(1.75)| \approx |10.0706 - 11.7632| = 1.6926.$$

The error bound is satisfies at every x tested above.

Problem 2.

Solution.

The exact integral is $\int_0^{\pi/3} \tan(x)\,\mathrm{d}x = \ln(\cos(x))|_{x=0}^{\pi/3} = \ln(2) \approx 0.693\,147\,18$.

Dividing the interval $[0, \frac{\pi}{3}]$ into $n = 3$ pieces, we have $h = \frac{\pi}{9}$ and

$$x_0 = 0, \mu_0 = \frac{\pi}{18}, x_1 = \frac{\pi}{9}, \mu_1 = \frac{\pi}{6}, x_2 = \frac{2\pi}{9}, \mu_2 = \frac{5\pi}{18}, x_3 = \frac{\pi}{3}.$$

x_k	$\tan(x_k)$
x_0	0
μ_0	0.176 326 98
x_1	0.363 970 23
μ_1	0.577 350 26
x_2	0.839 099 63
μ_2	1.191 753 59
x_3	1.732 050 81

The numerical integrals and their errors are given by

$$I_3^{\mathrm{LE}} = h\sum_{k=0}^{n-1} f(x_k) \approx 0.419\,950\,61 \quad \rightsquigarrow \quad e_3^{\mathrm{LE}} \approx 0.273\,196\,57$$

$$I_3^{\mathrm{TRAP}} = h\left(\frac{f(x_0)+f(x_n)}{2} + \sum_{k=1}^{n-1} f(x_k)\right) \approx 0.722\,250\,50 \quad \rightsquigarrow \quad e_3^{\mathrm{TRAP}} \approx 0.029\,103\,32$$

$$I_3^{\mathrm{MID}} = h\sum_{k=0}^{n-1} f(\mu_k) \approx 0.679\,083\,47 \quad \rightsquigarrow \quad e_3^{\mathrm{MID}} \approx 0.014\,063\,71.$$

Note that $f'(x) = 1 + \tan^2(x) = \sec^2(x)$ and $f''(x) = 2\tan(x)\sec^2(x)$ both of which are positive and increasing on $[0, \frac{\pi}{3}]$. We take $K_1 = \max_{x\in[0,\frac{\pi}{3}]} |f'(x)| = f(\frac{\pi}{3}) = 4$ and $K_2 = \max_{x\in[0,\frac{\pi}{3}]} |f''(x)| = 8\sqrt{3} \approx 13.856\,406\,46$.

The error bounds are therefore,

$$e_3^{\mathrm{LE}} \leq \frac{K_1}{2}(b-a)h \approx 0.731\,081\,81,$$

$$e_3^{\mathrm{TRAP}} \leq \frac{K_2}{12}(b-a)h^2 \approx 0.147\,337\,30,$$

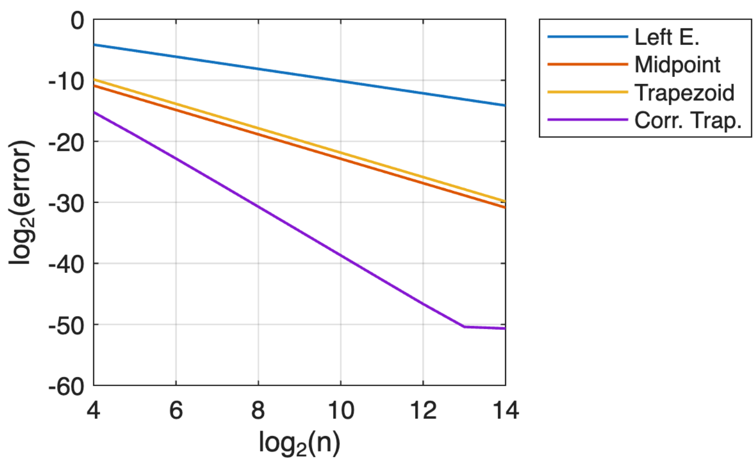
$$e_3^{\mathrm{MID}} \leq \frac{K_2}{24}(b-a)h^2 \approx 0.073\,668\,65.$$

Problem 3.

Solution. The numerical integrals are

n	I_n^{LE}	I_n^{MID}	I_n^{TRAP}	I_n^{CTRAP}
2^4	0.6375349	0.6926134	0.6942161	0.6931732
2^5	0.6650742	0.6930134	0.6934148	0.6931491
2^6	0.6790438	0.6931137	0.6932141	0.6931473
2^7	0.6860788	0.6931388	0.6931639	0.6931472
2^8	0.6896088	0.6931451	0.6931514	0.6931472
2^9	0.6913769	0.6931467	0.6931482	0.6931472
2^{10}	0.6922618	0.6931470	0.6931474	0.6931472
2^{11}	0.6927044	0.6931471	0.6931472	0.6931472
2^{12}	0.6929258	0.6931472	0.6931472	0.6931472
2^{13}	0.6930365	0.6931472	0.6931472	0.6931472
2^{14}	0.6930918	0.6931472	0.6931472	0.6931472

The error plot is as follows.



The slopes for the left endpoint method is -1, -2 for trapezoid and midpoint, and -4 for the corrected trapezoid method (until 2^{13}). The slope in the linear region corresponds to the order of the method.

Problem 4.

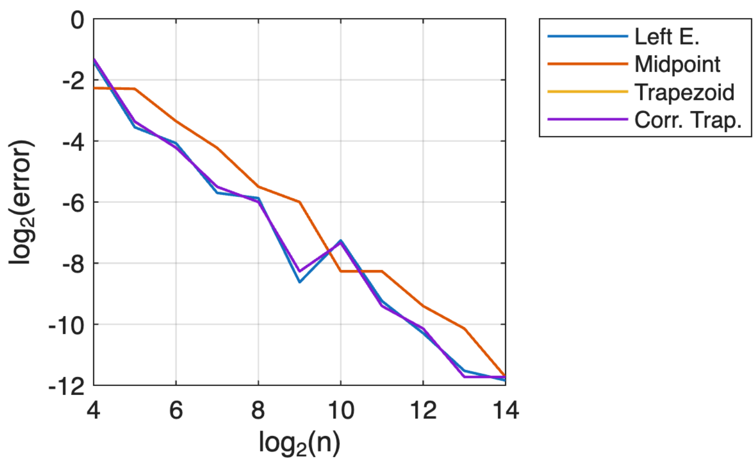
Example solution.

(a) `xrand` $= 0.31617962$.

(b)

n	I_n^{LE}	I_n^{MID}	I_n^{TRAP}	I_n^{CTRAP}
2^4	5.3418464	4.7565945	5.3644495	5.3662243
2^5	5.0492205	4.7601461	5.0605220	5.0610353
2^6	4.9046833	5.0618154	4.9103341	4.9104660
2^7	4.9832493	4.9107069	4.9860747	4.9861079
2^8	4.9469781	4.9861618	4.9483908	4.9483991
2^9	4.9665700	4.9484134	4.9672763	4.9672784
2^{10}	4.9574917	4.9672819	4.9578448	4.9578454
2^{11}	4.9623868	4.9672825	4.9625634	4.9625635
2^{12}	4.9648346	4.9625637	4.9649229	4.9649229
2^{13}	4.9636992	4.9649230	4.9637433	4.9637433
2^{14}	4.9643111	4.9637433	4.9643332	4.9643332

(c)



(d) All methods has a general downward trend of the error with a slope of roughly -1. The theoretical bounds we derived for integration methods are not valid because the integrand $f(x)$ is not differentiable and $x =$ `xrand`. Hence higher order methods do not exhibit significant advantage in this case.