Problem 1.

Solution.

$n = 1$:

Divide the interval $[0, T]$ into $n = 1$ step; $h = \frac{T}{n} = 2$.

Using the iteration formula for Euler's method, we have $x(0) = x_0 = 3$,

$$x_1 = x_0 + hf(t_0, x_0) = \boxed{5}.$$

$n = 2$:

Divide the interval $[0, T]$ into $n = 2$ steps; $h = \frac{T}{n} = 1$.

Using the iteration formula for Euler's method,

$$\begin{aligned} x_0 &= 3 \\ x_1 &= x_0 + hf(t_0, x_0) = 4 \\ x_2 &= x_1 + hf(t_1, x_1) = \boxed{5.5}. \end{aligned}$$

$n = 4$:

Divide the interval $[0, T]$ into $n = 4$ steps; $h = \frac{T}{n} = 0.5$.

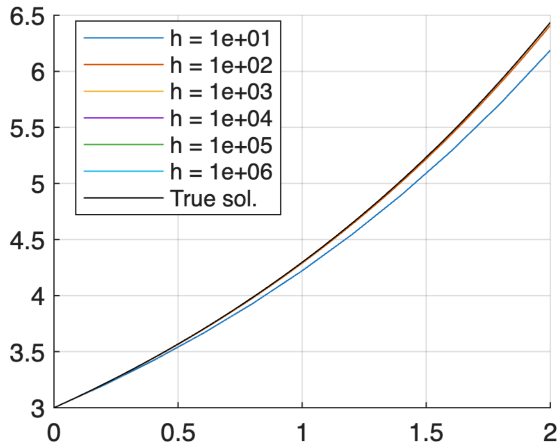
Using the iteration formula for Euler's method,

$$\begin{aligned} x_0 &= 3 \\ x_1 &= x_0 + hf(t_0, x_0) = 3.5 \\ x_2 &= x_1 + hf(t_1, x_1) = 4.125 \\ x_3 &= x_2 + hf(t_2, x_2) = 4.90625 \\ x_4 &= x_3 + hf(t_3, x_3) = \boxed{5.8828125}. \end{aligned}$$

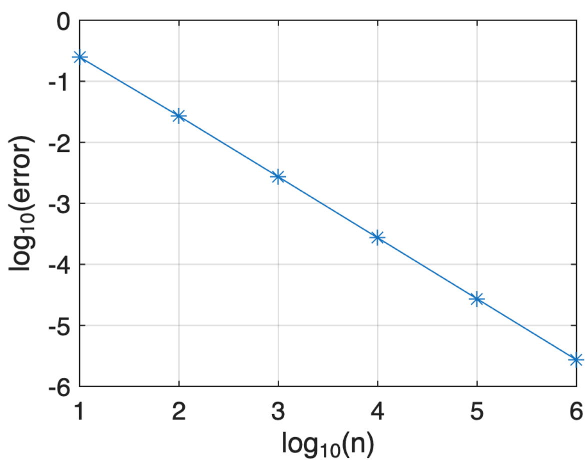
The true solution to the ODE is given by $\bar{x}(t) = 2e^{t/2} + 1$ and therefore $\bar{x}(T) = 2e + 1 = \boxed{6.4366}$.
The actual errors are $e_1 = 1.4366, e_2 = 0.9366$ and $e_4 = 0.5538$.

Problem 2.

Solution. As in Problem 1, the true solution is $\bar{x}(t) = 2e^{t/2} + 1$.



n	10^1	10^2	10^3	10^4	10^5	10^6
e_n	0.249079	0.026936	0.002716	0.000272	0.000027	0.000003

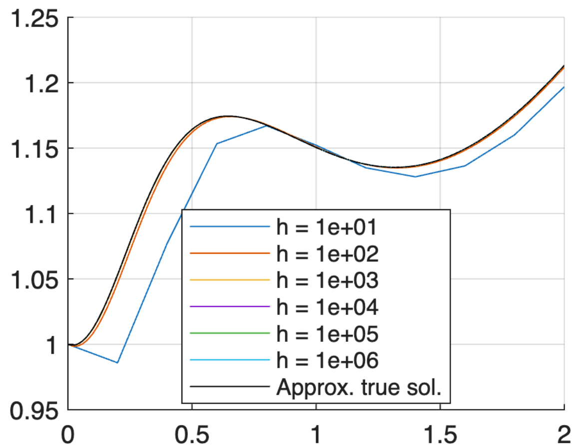


The slope in the error plot corrresponds to the order. In this plot, the slope is -1, which agrees with the Taylor polynomial based order estiamte $\mathcal{O}(n^{-1})$.

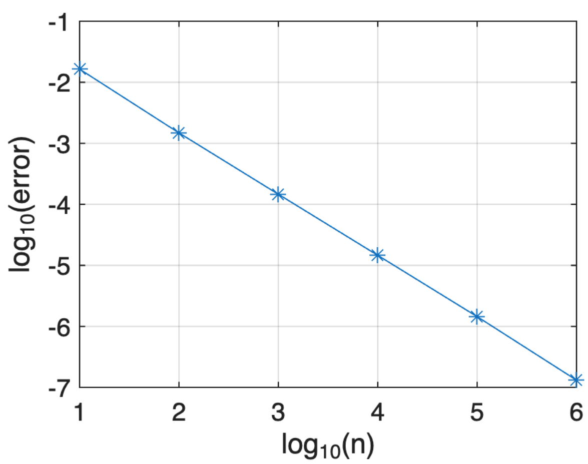
Problem 3.

Solution.

In this case, we do not have a closed form solution to the ODE. Instead, using the method from class, we take the solution generated by Euler's method with $n = 10^7$ steps as the "true solution".



n	10^1	10^2	10^3	10^4	10^5	10^6
e_n	0.016407	0.001479	0.000146	0.000015	1.44×10^{-6}	1.31×10^{-7}



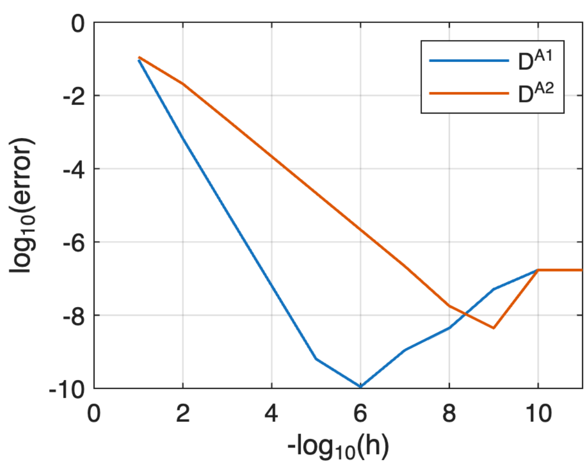
The slope in the error plot corrresponds to the order. In this plot, the slope is -1, which agrees with the Taylor polynomial based order estiamte $\mathcal{O}(n^{-1})$.

Problem 4.

Solution.

Using the same function from Problem 6 in HW2 ($f(x) = \cos(\pi x)\exp(-2x)$).

h	e_h^{A1}	e_h^{A2}
10^{-1}	9.46×10^{-2}	1.13×10^{-1}
10^{-2}	6.66×10^{-4}	2.07×10^{-2}
10^{-3}	6.39×10^{-6}	2.15×10^{-3}
10^{-4}	6.37×10^{-8}	2.15×10^{-4}
10^{-5}	6.34×10^{-10}	2.15×10^{-5}
10^{-6}	1.10×10^{-10}	2.15×10^{-6}
10^{-7}	1.11×10^{-9}	2.15×10^{-7}
10^{-8}	4.44×10^{-9}	1.78×10^{-8}
10^{-9}	5.11×10^{-8}	4.44×10^{-9}
10^{-10}	1.71×10^{-7}	1.71×10^{-7}
10^{-11}	1.71×10^{-7}	1.71×10^{-7}



From the log-log plot, one sees that D^{A1} is $\mathcal{O}(h^2)$, whereas D^{A2} is $\mathcal{O}(h)$. Therefore, we'd choose D^{A1}.