Problem 1.

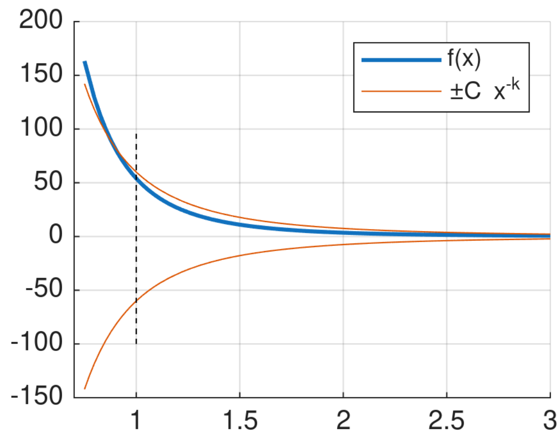
Solution. $f = \mathcal{O}(x^{-4})$.

$$|f(x)| = \left|\frac{57}{x^4} - \frac{3}{x^6}\right| \le \frac{57}{x^4} + \frac{3}{x^6}.$$

Since $x^{-6} < x^{-4}$ for $x > 1$. Choosing $D = 1$, we obtain

$$|f(x)| \le \frac{57}{x^4} + \frac{3}{x^4} = \frac{60}{x^4} \quad \text{for all}\, x > D.$$

We see that $C = 60$ is a valid choice.



(Vertical dashed line showing $D = 1$)

Problem 2.

Solution. We need to evaluate $f(x) = a_2 x + a_4 x^3 + a_6 x^5 + a_8 x^7$.

Using Horner's method, we may rearrange the polynomial as

$$f(x) = x(a_2 + x^2(a_4 + x^2(a_6 + a_8 x^2))).$$

Storing x^2 as a temporary variable (to save 2 multiplications), we have

```
GIVEN x
x_sq = x * x
tmp = a_8
tmp = a_6 + tmp * x_sq
tmp = a_4 + tmp * x_sq
tmp = a_2 + tmp * x_sq
tmp = x * tmp
RETURN tmp
```

This requires 3 additions and 5 multiplications, thus 8 FLOPs in total.

Problem 3.

Solution.

$$
\begin{aligned}
D_h(x_0) &= \frac{f(x_0+h)-f(x_0)}{h} = \frac{\tan(e^{-0.1})-\tan(e^{0})}{0.1} \approx \frac{1.2728-1.5574}{0.1} = -2.8465. \\
\hat{D}_h(x_0) &= \frac{f(x_0+h)-f(x_0-h)}{2h} = \frac{\tan(e^{-0.1})-\tan(e^{0.1})}{0.1} \approx \frac{1.2728-1.9901}{2\times 0.1} = -3.5870.
\end{aligned}
$$

The exact derivative is

$$
f'(x) = -e^{-x}\sec^2(e^{-x}) = -\frac{1}{e^x\cos^2(e^{-x})} \quad \rightsquigarrow \quad f'(x_0) = -\sec^2(1) = -\frac{1}{\cos^2(1)} = -3.4255.
$$

The actual errors are

$$
\begin{aligned}
e_h &= |D_h(x_0) - f'(x_0)| \approx 0.5790 \\
\hat{e}_h &= |\hat{D}_h(x_0) - f'(x_0)| \approx 0.1615.
\end{aligned}
$$

To compute the error bounds, first compute f'' and f''':

$$
\begin{aligned}
f''(x) &= e^{-x}\sec^2(e^{-x}) + 2e^{-2x}\tan(e^{-x})\sec^2(e^{-x}) \\
f'''(x) &= -e^{-x}\sec^2(e^{-x}) - 2e^{-3x}\sec^4(e^{-x}) - 6e^{-2x}\sec^2(e^{-x})\tan(e^{-x}) - 4e^{-3x}\sec^2(e^{-x})\tan^2(e^{-x}).
\end{aligned}
$$

We see that $f''(x) \geq 0$ on $[0,\bar{h}]$ and $f'''(x) \leq 0$ on $[-\bar{h},\bar{h}]$. Note that e^{-x} is decreasing, and $\sec(x), \tan(x)$ is increasing for $x \in [0, \frac{\pi}{2})$. Therefore, $\sec(e^{-x})$ and $\tan(e^{-x})$ are decreasing on $[-\bar{h},\bar{h}]$. It follows then $f''(x)$ is positive and decreasing on $[0,\bar{h}]$, and $f'''(x)$ is negative and increasing on $[-\bar{h},\bar{h}]$.

The error bounds are

$$
\begin{aligned}
e_h &\leq \frac{C_2(x_0,\bar{h})}{2}h = \frac{\max_{\zeta\in[0,0.25]}|f''(\zeta)|}{2}h = \frac{f''(0)}{2}h \approx \frac{14.0954}{2}\times 0.1 = 0.7048. \\
\hat{e}_h &\leq \frac{C_3(x_0,\bar{h})}{6}h^2 = \frac{\max_{\zeta\in[-0.25,0.25]}|f'''(\zeta)|}{6}h^2 = -\frac{f'''(-0.25)}{6}h^2 \approx \frac{2313.78}{6}\times 0.1^2 = 3.8563.
\end{aligned}
$$

Problem 4.

Solution. $f'(x) = \dfrac{1}{x}$, $f''(x) = -\dfrac{1}{x^2}$, $f'(x_0) = 0.5$.

Using D_h from Problem 3, we have

h	$D_h(x_0)$	$e_h(x_0)$
10^{-5}	0.499998750014274	1.25×10^{-6}
10^{-7}	0.499999986969257	1.30×10^{-8}
10^{-9}	0.500000041370185	4.14×10^{-8}

Since $|f''(x)|$ is decreasing, the maximum is always attained on the left endpoint x_0. The error bounds are

$$e_{10^{-5}} \leq \frac{C_2(x_0 = 2, \bar{h} = 10^{-5})}{2} h = \boxed{1.25 \times 10^{-6}} \quad \rightsquigarrow \quad \text{expect 5 correct digits,}$$

$$e_{10^{-7}} \leq \frac{C_2(x_0 = 2, \bar{h} = 10^{-7})}{2} h = \boxed{1.25 \times 10^{-8}} \quad \rightsquigarrow \quad \text{expect 7 correct digits,}$$

$$e_{10^{-9}} \leq \frac{C_2(x_0 = 2, \bar{h} = 10^{-9})}{2} h = \boxed{1.25 \times 10^{-10}} \quad \rightsquigarrow \quad \text{expect 9 correct digits.}$$

Problem 5.

Solution. Since the machine only keeps track of 15 or 16 digits, the number of reliable digits decreases as h gets too small.

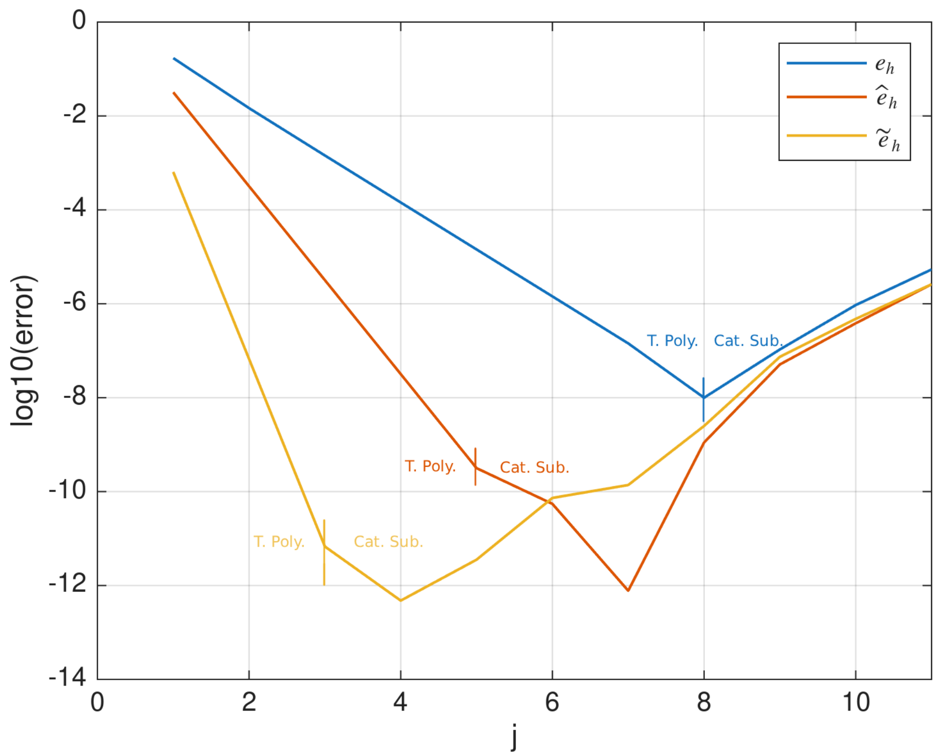
In this case, $f'(x)$ and $f(x)$ are of similar orders of magnitude, the best accuracy we could hope for is simply given by the ratio between the machine epsilon and h. That is, if $h = 10^{-k}$, the number of correct digits we could hope for would be $16 - k$ (or $15 - k$).

h	number of correct digits we could hope for
10^{-5}	11 (or 10)
10^{-7}	9 (or 8)
10^{-9}	7 (or 6)

Problem 6.

Solution. $f'(x_0) = -\frac{2+\pi}{\sqrt{2e}} \approx -2.20514$.

h	D_h	$\hat{D}_h$	$\tilde{D}_h$	e_h	$\hat{e}_h$	$\tilde{e}_h$
10^{-1}	-2.03437	-2.17314	-2.20578	1.7077×10^{-1}	3.1992×10^{-2}	6.4292×10^{-4}
10^{-2}	-2.19046	-2.20482	-2.20514	1.4674×10^{-2}	3.1828×10^{-4}	6.6650×10^{-8}
10^{-3}	-2.20370	-2.20513	-2.20514	1.4392×10^{-3}	3.1826×10^{-6}	6.7542×10^{-12}
10^{-4}	-2.20499	-2.20514	-2.20514	1.4364×10^{-4}	3.1827×10^{-8}	4.7651×10^{-13}
10^{-5}	-2.20512	-2.20514	-2.20514	1.4361×10^{-5}	3.1564×10^{-10}	3.5483×10^{-12}
10^{-6}	-2.20513	-2.20514	-2.20514	1.4361×10^{-6}	5.4738×10^{-11}	7.3242×10^{-11}
10^{-7}	-2.20514	-2.20514	-2.20514	1.4322×10^{-7}	7.7272×10^{-13}	1.3801×10^{-10}
10^{-8}	-2.20514	-2.20514	-2.20514	9.9912×10^{-9}	1.1110×10^{-9}	2.4988×10^{-9}
10^{-9}	-2.20514	-2.20514	-2.20514	1.0658×10^{-7}	5.1071×10^{-8}	7.4201×10^{-8}
10^{-10}	-2.20514	-2.20514	-2.20514	9.3925×10^{-7}	3.8414×10^{-7}	4.7666×10^{-7}
10^{-11}	-2.20514	-2.20514	-2.20514	5.3801×10^{-6}	2.6046×10^{-6}	2.6046×10^{-6}



The linear regions are where the Taylor polynomial based errors dominate, whereas the jagged regions are where catastrophic subtraction errors dominate. The slope in the linear region corresponds to the asymptotic order of error as h tends to 0. The e_h curve has slope -1 initially, which corresponded to D_h being $\mathcal{O}(h)$. Similarly, $\hat{e}_h$ has slope -2, which corresponded to $\hat{D}_h$ being $\mathcal{O}(h^2)$; $\tilde{e}_h$, -4, $\mathcal{O}(h^4)$.