Problem 1.

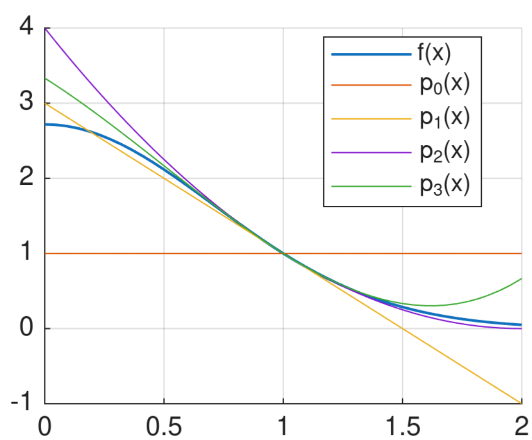
Solution.

n	$f^{(n)}(x)$	$f^{(n)}(x_0 = 1)$
0	e^{1-x^2}	1
1	$-2xe^{1-x^2}$	-2
2	$(4x^2-2)e^{1-x^2}$	2
3	$(12x-8x^3)e^{1-x^2}$	4

$$
\begin{aligned}
p_0(x) &= f^{(0)}(x_0)(x-x_0)^0 = 1\\
p_1(x) &= f^{(0)}(x_0)(x-x_0)^0 + f^{(1)}(x_0)(x-x_0)^1 = 1-2(x-1) = -2x+3\\
p_2(x) &= f^{(0)}(x_0)(x-x_0)^0 + f^{(1)}(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2\\
&= 1-2(x-1)+(x-1)^2 = x^2-4x+4\\
p_3(x) &= f^{(0)}(x_0)(x-x_0)^0 + f^{(1)}(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3\\
&= 1-2(x-1)+(x-1)^2+\frac{2}{3}(x-1)^3 = \frac{2}{3}x^3 - x^2 - 2x + \frac{10}{3}.
\end{aligned}
$$

Problem 2.

Solution.



Problem 3.

Solution.

n	$f^{(n)}(x)$	$f^{(n)}(x_0 = \frac{1}{4})$
0	$x - x^3$	$\frac{15}{64}$
1	$1 - 3x^2$	$\frac{13}{16}$

$$p_1(x) = f^{(0)}(x_0)(x - x_0)^0 + f^{(1)}(x_0)(x - x_0)^1 = \frac{15}{64} + \frac{13}{16}(x - \frac{1}{4}) = \frac{13}{16}x + \frac{1}{32}.$$

$$\begin{aligned} |R(x)| = \quad & |f(x) - p_1(x)| = \left|-x^3 + \frac{3}{16}x - \frac{1}{32}\right| \\ \rightsquigarrow \quad & |f(0) - p(0)| = \frac{1}{32} \\ & |f(1) - p(1)| = \frac{27}{32}. \end{aligned}$$

To find maximum actual error (i.e. $\max_{x\in[-1,1]} |R(x)|$), we check the critical points of $R(x)$ and the endpoints of the interval (in this case -1 and 1). The critical points of $R(x)$ is given by

$$R'(x) = 0 \quad \rightsquigarrow \quad \frac{3}{16} - 3x^2 = 0 \quad \rightsquigarrow \quad x = \pm\frac{1}{4}.$$

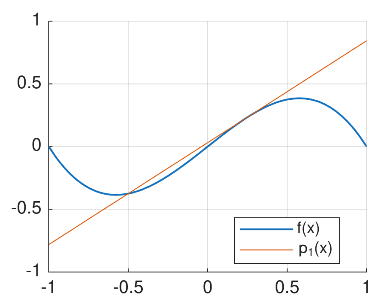
$$\begin{aligned} |R(-1)| = R(-1) &= \frac{25}{32} \\ |R(-1/4)| = -R(-1/4) &= \frac{1}{16} \\ |R(1/4)| = R(1/4) &= 0 \\ |R(1)| = -R(1) &= \frac{27}{32}. \end{aligned}$$

Hence the maximum actual error is $\frac{27}{32} = 0.84375$ and occurs at $x = 1$.

The minimum actual error is always 0, which is attained at x_0 (provided x_0 is in the interval of interest). We check if there are other places where the actual error is 0.

$$f(x) - p_1(x) = 0 \quad \rightsquigarrow \quad x = -\frac{1}{2}, \frac{1}{4}.$$

We see that the minimum actual error is 0, which occurs at $-\frac{1}{2}$ and $\frac{1}{4}$.



Problem 4.

Solution.

n	$f^{(n)}(x)$	$f^{(n)}(x_0=0)$
0	xe^x	0
1	$(1+x)e^x$	1
2	$(2+x)e^x$	2
3	$(3+x)e^x$	

$$p_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = x + x^2.$$

We have

$$B_2(x) = \frac{|x-0|^3}{3!} \max_{\zeta \in \overline{0,x}} |f'''(\zeta)| = \frac{|x|^3}{3!} \max_{\zeta \in \overline{0,x}} (3+\zeta)e^{\zeta} = \frac{|x|^3}{3!}(3+\max(0,x))e^{\max(0,x)},$$

where the maximum over ζ is attained at the right endpoint of $\overline{0,x}$ because $f'''(\zeta)$, being a product of two positive increasing functions, is a positive increasing function on $[-1,1]$.

To find the maximum of $B_2(x)$ over $[-1,1]$, it suffices to find the maximum of $B_2(-1)$ and $B_2(1)$ (this part does not need additional justification).

$$B_2(-1) = \frac{1}{3!} \cdot 3 = \frac{1}{2}, \quad B_2(1) = \frac{1}{3!} \cdot (4e) \approx 1.812188.$$

Hence the bound on the errors over $[-1,1]$, i.e. the maximum of $B_2(x)$ over $[-1,1]$, is 1.812188.

Actual errors: $|f(-1) - p_2(-1)| = \frac{1}{e} \approx 0.367879$, $|f(1) - p_2(1)| = e - 2 \approx 0.718282$.

Comparing to the error bounds: at $x = -1$, the error bound is about 0.132121 larger; at $x = 1$, the error bound is about 1.09391 larger.

The error bound is better on the left side ($x = -1$) because f''' (and hence f itself) is more flat (i.e. closer to a constant function) compared to the right side. Hence the error bound is closer to the actual error (cf. the exact remainder term in Theorem 1.1).

Problem 5.

Solution. We ensure that $B_n(x)$ is less than the desired error over the interval of interest. From Problem 4, we see that the n-th order derivative of f is given by

$$f^{(n)}(x) = (n+x)e^x.$$

Hence $f^{(n)}$ is increasing on $[-1,1]$ for all n (cf. the solution to Problem 4) and

$$\begin{aligned} B_n(-1) &= \frac{1^{n+1}}{(n+1)!} \max_{\zeta \in [-1,0]} |(n+1+\zeta)e^{\zeta}| = \frac{n+1}{(n+1)!} = \frac{1}{n!} \\ B_n(1) &= \frac{1^{n+1}}{(n+1)!} \max_{\zeta \in [0,1]} |(n+1+\zeta)e^{\zeta}| = \frac{(n+2)e}{(n+1)!}. \end{aligned}$$

Comparing $B_n(-1), B_n(1)$, we see that the only difference is the $\max_\zeta$ term, which is always greater for $B_n(1)$ than for $B_n(-1)$.

n	$B_n(-1)$	$B_n(1)$
0	1	5.4365636569
1	1	4.0774227427
2	0.5	1.8121878856
3	0.1666666667	0.5663087143
4	0.0416666667	0.1359140914
5	0.0083333333	0.0264277400
6	0.0013888889	0.0043147331
7	0.0001984127	0.0006067593
8	0.0000248016	0.0000749086
9	0.0000027557	0.0000082399

Sub-problem 1: $[-1,1]$, 10^{-2}.

Since the maximum of B_n over $[-1,1]$ is the greater of $B_n(-1)$ and $B_n(1)$, it suffices to look at just $B_n(1)$. The minimum n is 6.

Sub-problem 2: $[-1,1]$, 10^{-5}.

Again examining $B_n(1)$, the minimum n is 9.

Sub-problem 3: $[-1,0]$, 10^{-2}.

The maximum of B_n over $[-1,0]$ is the greater of $B_n(-1)$ and $B_n(0) = 0$. Hence by examining $B_n(-1)$, we see that the the minimum n is 5.

Problem 6.

Solution.

Pick $x_0 = \frac{1}{2}$ (midpoint of $[0,1]$).

The derivatives is computed as Problem 5. Then

$$B_n(0) = \frac{|0-\frac{1}{2}|^{n+1}}{(n+1)!} \max_{\zeta\in[0,\frac{1}{2}]} |(n+1+\zeta)e^{\zeta}| = \frac{(n+\frac{3}{2})e^{\frac{1}{2}}}{(n+1)!\cdot 2^{n+1}}$$
$$B_n(1) = \frac{|1-\frac{1}{2}|^{n+1}}{(n+1)!} \max_{\zeta\in[0,\frac{1}{2}]} |(n+1+\zeta)e^{\zeta}| = \frac{(n+2)e}{(n+1)!\cdot 2^{n+1}}.$$

Again, $B_n(1) > B_n(0)$ for all n. Therefore, it suffices to just consider $B_n(1)$.

n	$B_n(0)$	$B_n(1)$
0	1.2365409530	2.7182818285
1	0.5152253971	1.0193556857
2	0.1202192593	0.2265234857
3	0.0193209524	0.0353942946
4	0.0023614497	0.0042473154

We see that the the minimum order is $n = 4$.

Problem 7.

Solution. $f(x) = \mathcal{O}(x^2)$.

For $0 < x < 1 = \delta$, we have $x^3 < x^2$ and $x^5 < x^2$. Therefore,

$$|f(x)| = |x^5 + 3x^3 - 2x^2| \leq x^5 + 3x^3 + 2x^2 = x^2 + 3x^2 + 2x^2 = \underbrace{6}_{C} x^2 \quad \text{for all } x \in (0, \delta).$$

That is, $C = 6$, $n = 2$, $\delta = 1$.

The envelope is indicated in the plot below by the yellow curves.