MAE 107 Study Problems

1. Use Newton’s method to solve $3e^{-x} = 2x$ by hand (you may use a calculator) starting from $x_0 = 11$. What is your chosen function $f(x)$? Indicate $x_n$, $f(x_n)$ and $f'(x_n)$ at each step. Stop at the first $n$ such that $|f(x_n)| \leq 10^{-5}$.

We choose $f(x) = 3e^{-x} - 2x$. According to Newton’s method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$f(x_n)$</th>
<th>$f'(x_n)$</th>
<th>$x_{n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11</td>
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<td>-2.00005</td>
<td>0.00030</td>
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<td>2.99850</td>
<td>-4.99910</td>
<td>0.60011</td>
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<tr>
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<td>0.44604</td>
<td>-3.64625</td>
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<tr>
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<td>0.01183</td>
<td>-3.45670</td>
<td>0.72586</td>
</tr>
<tr>
<td>4</td>
<td>0.72586</td>
<td>0.000085</td>
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<td></td>
</tr>
</tbody>
</table>

Thus $x_4 = 0.72586$ is a good enough approximation of the root.

2. Apply the Newton’s method for $n > 1$ dimensions to find an approximate solution of the system

$$x^2 + y^2 = 7/4$$
$$3y = x^2$$

Start from $(x, y) = (2, 1)$ and perform 1 iteration by hand.

Using the notation $\mathbf{x} = (x \ y)^T$ we need to find the root of

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 7/4 \\ 3y - x^2 \end{pmatrix}$$

using the Newton’s method with initial conditions $\mathbf{x}_0 = (2 \ 1)^T$. We have

$$\mathbf{x}_{n+1} = \mathbf{x}_n - J^{-1}_f(\mathbf{x}_n)\mathbf{f}(\mathbf{x}_n)$$

where $J_f(\mathbf{x})$ is the Jacobian matrix given by

$$J_f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ -2x & 3 \end{pmatrix} \implies J_f^{-1}(\mathbf{x}) = \frac{1}{6x + 4xy} \begin{pmatrix} 3 & -2y \\ 2x & 2x \end{pmatrix}$$

Therefore

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{20} \begin{pmatrix} 3 & -2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 13/4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1.4125 \\ 0.5500 \end{pmatrix}$$
3. Use Lagrange interpolation to find the second order polynomial through \((x_0, y_0) = (1, 4), (x_1, y_1) = (3, 2)\) and \((x_2, y_2) = (7, -8)\). Reduce your solution to the form \(y = c_0 + c_1x + c_2x^2\).

The \(n\)th order Lagrange polynomial is given by

\[
p_n(x) = \sum_{i=0}^{n} \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} y_i
\]

\[
p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2
\]

\[
= \frac{(x - 3)(x - 7)}{(1 - 3)(1 - 7)} \times 4 + \frac{(x - 1)(x - 7)}{(3 - 1)(3 - 7)} \times 2 - \frac{(x - 1)(x - 3)}{(7 - 1)(7 - 3)} \times 8
\]

\[
= -\frac{1}{4}x^2 + \frac{17}{4}
\]

4. Use the secant method to solve \(\arctan(x/2) = -x - 1\) by hand (you may use a calculator) starting from \(x_0 = 2\) and \(x_1 = 4\). What is your chosen function \(f(x)\)? Indicate \(x_n\) and \(f(x_n)\). Stop at the first \(n\) such that \(|f(x_n)| \leq 10^{-4}\).

We choose \(f(x) = \arctan(x/2) + x + 1\). According to the secant method

\[
x_{k+1} = x_k - \frac{f(x_k)}{f(x_k) - f(x_{k-1})} \left( x_k - x_{k-1} \right)
\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x_{n-1})</th>
<th>(x_n)</th>
<th>(f(x_n))</th>
<th>(x_{n+1})</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>-0.63586</td>
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<td>-0.67587</td>
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<tr>
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<tr>
<td>5</td>
<td>-0.67587</td>
<td>-0.67466</td>
<td>0.0000</td>
<td></td>
</tr>
</tbody>
</table>

Thus \(x_5 = -0.67466\) is a good enough approximation of the root.

5. Suppose you are running the fixed point method to solve \(x = g(x)\), where you know \(|g'(x)| \leq 0.2\) for all \(x\). Starting from \(x_0 = 3\), you obtain \(x_1 = -1.702\). What is the minimum \(n\) such that you can guarantee the error in \(x_n\) will be less than \(10^{-4}\).

By Theorem 3.5 of the textbook, if \(\bar{x}\) is the true solution of \(x = g(x)\) then an upper bound on the error of the fixed-point method is given by

\[
|\bar{x} - x_n| \leq \frac{\gamma^n}{1 - \gamma} |x_1 - x_0|
\]

where \(\gamma\) is an upper bound on \(|g'(x)|\). In this case, we may use \(\gamma = 0.2\). In order to ensure that the error is no greater than \(10^{-4}\)

\[
\frac{(0.2)^n}{0.8} | -1.702 - 3 | \leq 10^{-4} \implies n \geq -\frac{\log(5.8755 \times 10^4)}{\log(0.2)} \approx 6.8232 \implies n = 7
\]

6. We have taken data on a system that we expect to behave according to

\[
x(t) = a_0 + a_1 t
\]
where the coefficients $a_0$ and $a_1$ are unknown. The data is

\[(t_0, x_0) = (2, 4)\]
\[(t_1, x_1) = (3, 5)\]
\[(t_2, x_2) = (5, 7)\]
\[(t_3, x_3) = (7, 10)\]
\[(t_4, x_4) = (9, 15)\]

Use least-squares to estimate $a_0$ and $a_1$ (you are free to use a calculator).

There are $n = 5$ data points. By least-squares we have

\[a_1 = \frac{n \sum_{k=0}^{n-1} x_k y_k - \left(\sum_{k=0}^{n-1} x_k\right) \left(\sum_{k=0}^{n-1} y_k\right)}{n \sum_{k=0}^{n-1} x_k^2 - \left(\sum_{k=0}^{n-1} x_k\right)^2} = \frac{5 \times 263 - 26 \times 41}{5 \times 168 - 26^2} = \frac{249}{164} = 1.5183 \ldots\]

\[a_0 = \frac{\left(\sum_{k=0}^{n-1} x_k^2\right) \left(\sum_{k=0}^{n-1} y_k\right) - \left(\sum_{k=0}^{n-1} x_k\right) \left(\sum_{k=0}^{n-1} x_k y_k\right)}{n \sum_{k=0}^{n-1} x_k^2 - \left(\sum_{k=0}^{n-1} x_k\right)^2} = 0.3049 \ldots\]

7. Approximately compute by hand

\[\int_0^{\pi/4} \tan^3(\theta) d\theta\]

by the trapezoid rule, the midpoint rule and the corrected trapezoid rule with $n = 3$ steps.

We have $h = (\pi/4 - 0)/3 = \pi/12$.

\[I_{\text{TRAP}} = \frac{h}{2} \left( f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) \right)\]
\[= \frac{\pi}{24} \left( \tan^3(0) + 2\tan^3(\pi/12) + 2\tan^3(\pi/6) + \tan^3(\pi/4) \right)\]
\[= 0.1863\]

\[I_{\text{TRAPC}} = I_{\text{TRAP}} - \frac{h}{24} \left( 3f(x_3) - 4f(x_2) + f(x_1) + 3f(x_0) - 4f(x_1) + f(x_2) \right)\]
\[= I_{\text{TRAP}} - \frac{h}{8} \left( f(x_0) - f(x_1) - f(x_2) + f(x_3) \right)\]
\[= I_{\text{TRAP}} - \frac{\pi}{96} \left( \tan^3(0) - \tan^3(\pi/12) - \tan^3(\pi/6) + \tan^3(\pi/4) \right)\]
\[= 0.1605\]

\[I_{\text{MID}} = h \left( f(\pi/24) + f(\pi/8) + f(5\pi/24) \right) = 0.1375\]