MAE 107 Homework 5 Solutions

1. We evaluate the integral \( I = \int_0^{1/2} \sin(\pi x) \cos(\pi x) \, dx \) using the left-endpoint rectangle rule. Here \( a = 0, \ b = 1/2 \) and \( n = 3 \) and \( f(x) = \frac{1}{2} \sin(2\pi x) \). For a uniform grid \( h = (b-a)/n = (1/2-0)/3 = 1/6 \). So

\[
\begin{align*}
    x_0 &= a = 0 \\
    x_1 &= a + h = 0 + 1/6 = 1/6 \\
    x_2 &= a + 2h = 0 + 2 \times (1/6) = 1/3 \\
    x_3 &= a + 3h = b = 1/2
\end{align*}
\]

So by the left-endpoint rule we get

\[
I_{\text{LEP}}(f) = h (f(x_0) + f(x_1) + f(x_2))
\]

\[
= \frac{1}{12} \left( \sin(2\pi \times 0) + \sin(2\pi/6) + \sin(2\pi/3) \right)
\]

\[
= \frac{1}{12} \left( 0 + \sqrt{3}/2 + \sqrt{3}/2 \right) = \frac{\sqrt{3}}{12}
\]

The actual value of the integral is \( I = \frac{1}{2} \times \frac{1}{2\pi} (1 - \cos \pi) = \frac{1}{4\pi} \times 2 = \frac{1}{2\pi} \). Therefore, the error is \( e = |1/2\pi - \sqrt{3}/2| = 0.0148 \). The error bound for left-endpoint rule is

\[
e_{\text{LEP}} \leq \frac{(b-a)h}{2} \max_{\xi \in [a,b]} |f'(\xi)|
\]

\[
= \frac{1/2 - 0}{2} \times \frac{1}{6} \max_{\xi \in [0,1/2]} |\pi \cos(2\pi x)| = \frac{\pi}{24}
\]

In order to make the error less than \( 10^{-4} \) we need

\[
\frac{(b-a)h}{2} \times \pi \leq 10^{-4} \implies n \geq \frac{(b-a)^2}{2} \times 10^4 \times \pi = \frac{1}{8} \times 10^4 \times \pi \approx 3926.99 \ldots
\]

Therefore, since \( n \in \mathbb{N} \) we have \( n = 3927 \).

2. By the trapezoidal rule we have

\[
I_{\text{TRAP}}(f) = \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3))
\]

\[
= \frac{1}{24} \left( \sin(2\pi \times 0) + 2 \sin(2\pi/6) + 2 \sin(2\pi/3) + \sin(2\pi/2) \right)
\]

\[
= \frac{1}{12} \left( \sqrt{3}/2 + \sqrt{3}/2 \right) = \frac{\sqrt{3}}{12}
\]

The actual value of the integral is \( I = \frac{1}{2} \times \frac{1}{2\pi} (1 - \cos \pi) = \frac{1}{4\pi} \times 2 = \frac{1}{2\pi} \). Therefore, the error is \( e = |1/2\pi - \sqrt{3}/2| = 0.0148 \). The error bound for the trapezoidal rule is

\[
e_{\text{TRAP}} \leq \frac{(b-a)h^2}{12} \max_{\xi \in [a,b]} |f''(\xi)|
\]

\[
= \frac{1/2 - 0}{12} \times \frac{1}{36} \max_{\xi \in [0,1/2]} |2\pi^2 \sin(2\pi x)| = \frac{\pi^2}{432}
\]
In order to make the error less than $10^{-4}$ we need

$$\frac{(b-a)h^2}{12} \times 2\pi^2 \leq 10^{-4} \implies n^2 \geq \frac{(b-a)^3}{12} \times 10^4 \times 2\pi^2 = \frac{1}{96} \times 10^4 \times 2\pi^2 \implies n \gtrsim 45.3449\ldots$$

Therefore, since $n \in \mathbb{N}$ we have $n = 46$.

3.

```matlab
% Objective: Evaluate the left-endpoint and trapezoidal approximations
% of the integral of \(\sin(pi \cdot x) \cdot \cos(pi \cdot x)\) and plot the error versus \(n\)
% Error vectors
e_lep = zeros(11, 1);
e_trap = zeros(11, 1);
% Variables to store the approximations
l = zeros(11, 1);
t = zeros(11, 1);
for i = 1 : 11
  % Computing the approximations
  l(i) = lep(@(x)sin(pi*x).*cos(pi*x), 0, 1/2, 2^i);
t(i) = trap(@(x)sin(pi*x).*cos(pi*x), 0, 1/2, 2^i);
end
% Computing the errors
e_lep = abs(l-1/(2*pi));
e_trap = abs(t-1/(2*pi));
% Log-log plot of errors versus \(n\)
plot(log10(2.^(1:11)), log10(e_lep), '-o')
hold on
plot(log10(2.^(1:11)), log10(e_trap), '-o')
xlabel('\$\log_{10} n\$', 'Interpreter', 'Latex')
ylabel('\$\log_{10} e\$', 'Interpreter', 'Latex')
title('Log-log plot of errors versus \(n\)')
legend('left-endpoint', 'trapezoidal')
```

```matlab
% Objective: Evaluate the function \(\sin(pi \cdot x) \cdot \cos(pi \cdot x)\)
function y = sin(pi*x).*cos(pi*x)
% Inputs: \(x\)
% Ouptuts: \(\sin(pi \cdot x) \cdot \cos(pi \cdot x)\)
y = sin(pi*x).*cos(pi*x);
end
```

```matlab
% Objective: Evaluate the left-endpoint approximation of the integral of \(f\) on \([a, b]\)
function y = lep(f, a, b, n)
% Inputs:
% \(f\) = function to be integrated
% \(a\) = left endpoint of interval
% \(b\) = right endpoint of interval
% \(n\) = number of partitions of interval
% Ouptuts:
% \(y\) = left-endpoint approximation
x = linspace(a, b, n+1);
h = (b-a)/n;
y = h*sum(f(x(1:n)));
end
```
We note that the slope of both plots is $-2$ which is expected for the trapezoidal rule. However, the left-endpoint rule performs better than the expected $O(n^{-1})$ order bound, which we note is only a bound on the errors. The unexpectedly good performance is due to the symmetry of the integrand. More specifically, the underestimate of the function values by the left-endpoint method on the first half of the interval is compensated by the overestimate of the function values on the second half of the interval (since the function is initially increasing and then decreasing).

4.

1 % Objective: Solve $Ax = b$ for tridiagonal $A$
2 function $x = $ tridiag($d$, 1, $u$, $f$)
3 % Inputs:
4 % $d =$ diagonal elements of $A$
5 % $l =$ lower-diagonal elements of $A$
6 % $u =$ upper-diagonal elements of $A$
7 % $f =$ $b$
8 % Outputs:
9 % $x =$ Solution to $Ax = b$
10 $n =$ size($d$, 1);
x = zeros(n,1);
% Elimination stage
for i = 2 : n
    % Note: l(i-1, :) since MATLAB indexing differs from the
    % notation in the textbook
    d(i, :) = d(i, :) - u(i-1, :) * l(i-1, :) / d(i-1, :);
    f(i, :) = f(i, :) - f(i-1, :) * l(i-1, :) / d(i-1, :);
end
% Backsolve stage
x(n, :) = f(n, :) / d(n, :);
for i = n-1 : -1 : 1
    x(i, :) = (f(i, :) - u(i, :) * x(i+1, :)) / d(i, :);
end
% Objective: Solve Ax = b for given A and b
d = [6; 8; 9; 8; 9; 4];
l = [3; 2; 4; -2; 2];
u = [4; 3; 4; -3; -1];
b = [2; 4; 6; 8; 10; 12];
x = tridiag(d, l, u, b);
disp((diag(d) + diag(l, -1) + diag(u, 1)) * x - b)

>> ans =
  1.0000e-14 *
     0
     0
     0
     0
-0.0888
  0.1776
     0