1. Taylor polynomial of order 4 around $x_0 = -1$ for $h(x) = 2f(x) - 3g(x)$

$$p_4^h(x_0) = h(x_0) + \frac{h^{(1)}(x_0)}{1!}(x - x_0) + \frac{h^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{h^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{h^{(4)}(x_0)}{4!}(x - x_0)^4$$

By definition of $h(x)$ and linearity of differentiation, we have

$$h(x_0) = 2f(x_0) - 3g(x_0)$$
$$h^{(k)}(x_0) = 2f^{(k)}(x_0) - 3g^{(k)}(x_0)$$

Therefore, we have

$$p_4^h(x_0) = \sum_{k=0}^{4} \frac{2f^{(k)}(x_0) - 3g^{(k)}(x_0)}{k!} (x - x_0)^k$$

$$= 2p_4^f(x_0) - 3p_4^g(x_0)$$
$$= 2 \cdot \left[ \frac{3}{2}(x + 1) - 3(x + 1)^3 \right] - 3 \cdot \left[ 8 + 2(x + 1)^2 + (x + 1)^4 \right]$$

$$= -24 + 3(x + 1) - 6(x + 1)^2 - 6(x + 1)^3 - 3(x + 1)^4$$

2. Asymptotic order of $f(x) = 3x + 11x^2 - 2x^4$ as $x \downarrow 0$

Choose $\delta = 1$. For $|x| < \delta$,

$$|f(x)| = |3x + 11x^2 - 2x^4|$$
$$\leq |3x| + |11x^2| + |2x^4|$$
$$\leq |3x| + |11x| + |2x|$$
$$\leq 16|x|$$

Therefore, $f(x) = \mathcal{O}(x)$ as $x \downarrow 0$, with $C = 16$, $\beta(x) = x$, $\delta = 1$. 

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3. Show that the asymptotic order of \( f(x) = 5x^3 \) as \( x \downarrow 0 \) is not \( O(x^4) \)

We will prove this by contradiction. Suppose \( f(x) = O(x^4) \). Then there exists \( C < \infty \) and \( \delta > 0 \) such that for all \( x \in (0, \delta) \)

\[
5|x^3| < C|x^4|
\]

Divide both sides by \( x^4 \), this becomes

\[
\frac{5}{x} < C
\]

for all \( x \in (0, \delta) \). However, \( 5/x \to \infty \) as \( x \) gets close to 0, so the inequality cannot hold for any finite \( C \) and \( \delta > 0 \).

4. Asymptotic order of \( g(x) = \tan(x^2) \) as \( x \downarrow 0 \)

Write down the \( k \)-th derivatives of \( \tan(x^2) \) at \( x_0 = 0 \):

\[
\begin{align*}
    f^{(0)}(x) &= \tan(x^2); & f^{(0)}(x_0) &= 0 \\
    f^{(1)}(x) &= 2x \sec^2(x^2); & f^{(1)}(x_0) &= 0 \\
    f^{(2)}(x) &= 8x^2 \tan(x^2) \sec^2(x^2) + 2 \sec^2(x^2); & f^{(2)}(x_0) &= 2
\end{align*}
\]

The first nonzero coefficient is \( f^{(2)}(x_0) \). Therefore, we take \( n = 2 - 1 = 1 \) and write \( f(x) = p_1(x) + R_1(x) \). Take \( \delta = \pi/4 \). Then for \( x \in (0, \pi/4) \), there exists an \( \xi \in (0, x] \) such that:

\[
\tan(x^2) = f^{(0)}(x_0) + f^{(1)}(x_0) \cdot (x - x_0) + R_1(x)
\]

\[
= 0 + 0 + \frac{f^{(2)}(\xi)}{2!} x^2
\]

Therefore,

\[
|\tan(x^2)| \leq 0 + \left| \frac{f^{(2)}(\xi)}{2!} x^2 \right|
\]

\[
\leq 0 + \max_{\xi \in (0, \pi/4)} \left| \frac{f^{(2)}(\xi)}{2!} x^2 \right|
\]
Because $f^{(2)}(x)$ is increasing for $x \in (0, \pi/4)$, we have
\[
|\tan(x^2)| \leq \frac{f^{(2)}(\pi/4)}{2!} x^2 \\
\leq 4.13|x^2|
\]
Therefore, $\tan(x^2) = \mathcal{O}(x^2)$ as $x \downarrow 0$.

Alternative solution: Take $n = 2$ and $f(x) = p_2(x) + R_2(x)$. Take $\delta = \pi/4$. Then for $x \in (0, \pi/4)$, there exists an $\xi \in (0, x]$ such that:
\[
\tan(x^2) = f^{(0)}(x_0) + f^{(1)}(x_0) \cdot (x - x_0) + \frac{f^{(2)}(x_0)}{2!} (x-x_0)^2 + R_2(x) \\
= 0 + 0 + x^2 + \frac{f^{(3)}(\xi)}{3!} x^3
\]
Therefore,
\[
|\tan(x^2)| \leq |x^2| + \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right| \\
\leq |x^2| + \max_{\xi \in (0,\pi/4)} \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right|
\]
Since $f^{3}(x) = 16x^3 \sec^4(x^2) + 24x \sec^2(x^2) \tan(x^2) + 32x^3 \sec^2(x^2) \tan^2(x^2)$ is strictly increasing on $(0, \pi/4)$, we have
\[
|\tan(x^2)| \leq |x^2| + \left| \frac{f^{(3)}(\pi/4)}{3!} x^3 \right| \\
= |x^2| + \frac{49.31}{3!} |x^3| \\
\leq 9.22|x^2|
\]
Therefore, $\tan(x^2) = \mathcal{O}(x^2)$ as $x \downarrow 0$.

5. **Asymptotic order of** $f(x) = \frac{3}{x^2} - \frac{5}{x^4}$ **as** $x \to \infty$

Choose $R = 1$. For $x > R$, we have
\[
|f(x)| = \left| \frac{3}{x^2} - \frac{5}{x^4} \right| \\
\leq \left| \frac{3}{x^2} \right| + \left| \frac{5}{x^4} \right| \\
\leq \left| \frac{3}{x^2} \right| + \left| \frac{5}{x^4} \right| \\
\leq \left| \frac{8}{x^2} \right|
\]
Therefore, $f(x) = \mathcal{O}(x^{-2})$ as $x \to \infty$, with $C = 8$, $R = 1$, $\beta(x) = x^{-2}$.
6. Asymptotic order of \( f(x) = \frac{7}{x^2 + x^9} + \frac{3}{x^3 + x^5} \) as \( x \to \infty \)

Let \( R = 1 \). For \( x > R \), we have

\[
\begin{align*}
\left| \frac{7}{x^2 + x^9} \right| &\leq \left| \frac{7}{x^9} \right| \\
\left| \frac{3}{x^3 + x^5} \right| &\leq \left| \frac{3}{x^5} \right|
\end{align*}
\]

To find the tightest order, we choose \( x^{-9} \) and \( x^{-5} \)

\[
|f(x)| \leq \left| \frac{7}{x^2 + x^9} \right| + \left| \frac{3}{x^3 + x^5} \right|
\]

\[
\leq \frac{7}{x^9} + \frac{3}{x^5}
\]

\[
\leq \frac{7}{x^5} + \frac{3}{x^5}
\]

\[
\leq \frac{10}{x^5}
\]

Therefore, \( f(x) = \mathcal{O}(x^{-5}) \) as \( x \to \infty \).

7. Asymptotic order of \( f(x) = 3x^3 y(x) \) \( x \downarrow 0 \)

\( y(x) = \mathcal{O}(x^5) \) as \( x \downarrow 0 \), therefore \( \exists C > 0, \delta > 0 \) s.t.

\[
|y(x)| \leq C|x^5|
\]

for all \( x \in (0, \delta) \). Substitute in \( |f(x)| \),

\[
|f(x)| \leq |3x^3 y(x)|
\]

\[
\leq 3C|x^8|
\]

Therefore, \( f(x) = \mathcal{O}(x^8) \).