1. Taylor polynomial of order 4 around \( x_0 = -1 \) for \( h(x) = 2f(x) - 3g(x) \)

\[
p_h^4(x_0) = h(x_0) + \frac{h^{(1)}(x_0)}{1!}(x - x_0) + \frac{h^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{h^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{h^{(4)}(x_0)}{4!}(x - x_0)^4
\]

By definition of \( h(x) \) and linearity of differentiation, we have

\[
h(x_0) = 2f(x_0) - 3g(x_0)
\]

\[
h^{(k)}(x_0) = 2f^{(k)}(x_0) - 3g^{(k)}(x_0)
\]

Therefore, we have

\[
p_h^4(x_0) = \sum_{k=0}^{4} \frac{2f^{(k)}(x_0) - 3g^{(k)}(x_0)}{k!} (x - x_0)^k
\]

\[
= 2p_2^f(x_0) - 3p_2^g(x_0)
\]

\[
= 2 \cdot \left[ \frac{3}{2} (x + 1) - 3(x + 1)^3 \right] - 3 \cdot \left[ 8 + 2(x + 1)^2 + (x + 1)^4 \right]
\]

\[
= -24 + 3(x + 1) - 6(x + 1)^2 - 6(x + 1)^3 - 3(x + 1)^4
\]

2. Asymptotic order of \( f(x) = 3x + 11x^2 - 2x^4 \) as \( x \downarrow 0 \)

Choose \( \delta = 1 \). For \( |x| < \delta \),

\[
|f(x)| = |3x + 11x^2 - 2x^4| 
\]

\[
\leq |3x| + |11x^2| + |2x^4| 
\]

\[
\leq |3x| + |11x| + |2x| 
\]

\[
\leq 16|x|
\]

Therefore, \( f(x) = O(x) \) as \( x \downarrow 0 \), with \( C = 16, \beta(x) = x, \delta = 1 \).
3. Show that the asymptotic order of \( f(x) = 5x^3 \) as \( x \downarrow 0 \) is not \( O(x^4) \)

We will prove this by contradiction. Suppose \( f(x) = O(x^4) \). Then there exists \( C < \infty \) and \( \delta > 0 \) such that for all \( x \in (0, \delta) \)

\[
5|x^3| < C|x^4|
\]

Divide both sides by \( x^4 \), this becomes

\[
\frac{5}{x} < C
\]

for all \( x \in (0, \delta) \). However, \( 5/x \to \infty \) as \( x \) gets close to 0, so the inequality cannot hold for any finite \( C \) and \( \delta > 0 \).

4. Asymptotic order of \( g(x) = \tan(x^2) \) as \( x \downarrow 0 \)

Write down the \( k \)-th derivatives of \( \tan(x^2) \) at \( x_0 = 0 \):

\[
\begin{align*}
  f^{(0)}(x) &= \tan(x^2); \quad f^{(0)}(x_0) = 0 \\
  f^{(1)}(x) &= 2x \sec^2(x^2); \quad f^{(1)}(x_0) = 0 \\
  f^{(2)}(x) &= 8x^2 \tan(x^2) \sec^2(x^2) + 2 \sec^2(x^2); \quad f^{(2)}(x_0) = 2
\end{align*}
\]

The first nonzero coefficient is \( f^{(2)}(x_0) \). Therefore, we take \( n = 2 - 1 = 1 \) and write \( f(x) = p_1(x) + R_1(x) \). Take \( \delta = \pi/4 \). Then for \( x \in (0, \pi/4) \), there exists an \( \xi \in (0, x) \) such that:

\[
\tan(x^2) = f^{(0)}(x_0) + f^{(1)}(x_0) \cdot (x - x_0) + R_1(x)
\]
\[
= 0 + 0 + \frac{f^{(2)}(\xi)}{2!} x^2
\]

Therefore,

\[
|\tan(x^2)| \leq 0 + \left| \frac{f^{(2)}(\xi)}{2!} x^2 \right|
\]
\[
\leq 0 + \max_{\xi \in (0, \pi/4)} \left| \frac{f^{(2)}(\xi)}{2!} x^2 \right|
\]
Because $f^{(2)}(x)$ is increasing for $x \in (0, \pi/4)$, we have

$$|\tan(x^2)| \leq \left| \frac{f^{(2)}(\pi/4)}{2!} x^2 \right| \leq 4.13|x^2|$$

Therefore, $\tan(x^2) = O(x^2)$ as $x \downarrow 0$.

**Alternative solution**

Take $n = 2$ and $f(x) = p_2(x) + R_2(x)$. Take $\delta = \pi/4$. Then for $x \in (0, \pi/4)$, there exists an $\xi \in (0, x]$ such that:

$$\tan(x^2) = f^{(0)}(x_0) + f^{(1)}(x_0) \cdot (x - x_0) + \frac{f^{(2)}(x_0)}{2!} \cdot (x - x_0)^2 + R_2(x)$$

$$= 0 + 0 + \frac{f^{(3)}(\xi)}{3!} x^3$$

Therefore,

$$|\tan(x^2)| \leq |x^2| + \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right|$$

$$\leq |x^2| + \max_{\xi \in (0, \pi/4)} \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right|$$

Since $f^3(x) = 16x^3 \sec^4(x^2) + 24x \sec^2(x^2) \tan(x^2) + 32x^3 \sec^2(x^2) \tan^2(x^2)$ is strictly increasing on $(0, \pi/4)$, we have

$$|\tan(x^2)| \leq |x^2| + \left| \frac{f^{(3)}(\pi/4)}{3!} x^3 \right|$$

$$= |x^2| + \frac{49.31}{3!} |x^3|$$

$$\leq 9.22|x^2|$$

Therefore, $\tan(x^2) = O(x^2)$ as $x \downarrow 0$.

5. **Asymptotic order of $f(x) = \frac{3}{x^2} - \frac{5}{x^4}$ as $x \to \infty$**

Choose $R = 1$. For $x > R$, we have

$$|f(x)| = \left| \frac{3}{x^2} - \frac{5}{x^4} \right|$$

$$\leq \frac{3}{x^2} + \frac{5}{x^4}$$

$$\leq \frac{3}{x^2} + \frac{5}{x^2}$$

$$\leq \frac{8}{x^2}$$

Therefore, $f(x) = O(x^{-2})$ as $x \to \infty$, with $C = 8$, $R = 1$, $\beta(x) = x^{-2}$. 


6. Asymptotic order of \( f(x) = \frac{7}{x^2 + x^9} + \frac{3}{x^3 + x^5} \) as \( x \to \infty \)

Let \( R = 1 \). For \( x > R \), we have

\[
\begin{align*}
\left| \frac{7}{x^2 + x^9} \right| &\leq \left| \frac{7}{x^9} \right| \\
\left| \frac{3}{x^3 + x^5} \right| &\leq \left| \frac{3}{x^5} \right|
\end{align*}
\]

To find the tightest order, we choose \( x^{-9} \) and \( x^{-5} \)

\[
|f(x)| \leq \left| \frac{7}{x^2 + x^9} \right| + \left| \frac{3}{x^3 + x^5} \right|
\leq \frac{7}{x^9} + \frac{3}{x^5}
\leq \frac{7}{x^5} + \frac{3}{x^5}
\leq \frac{10}{x^5}
\]

Therefore, \( f(x) = \mathcal{O}(x^{-5}) \) as \( x \to \infty \).

7. Asymptotic order of \( f(x) = 3x^3y(x) \) \( x \downarrow 0 \)

\( y(x) = \mathcal{O}(x^5) \) as \( x \downarrow 0 \), therefore \( \exists C > 0, \delta > 0 \) s.t.

\[
|y(x)| \leq C|x^5|
\]

for all \( x \in (0, \delta) \). Substitute in \( |f(x)| \),

\[
|f(x)| \leq |3x^3y(x)|
\leq 3C|x^8|
\]

Therefore, \( f(x) = \mathcal{O}(x^8) \).