MAE 107 Homework #1 Solutions

1. Constructing the 0th, 1st and 2nd order of Taylor Polynomials:

\( p_0(x) = f(x_0) \)
\( p_1(x) = f(x_0) + f'(x_0)(x - x_0) \)
\( p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \)

\[ f(x) = \cos\left(\pi + \frac{x}{2}\right) = -\cos\left(\frac{x}{2}\right) \]
\[ f'(x) = \frac{1}{2}\sin\left(\frac{x}{2}\right) \]
\[ f''(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right) \]

With \( x_0 = -\frac{\pi}{2} \)

\[ f(x_0) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \]
\[ f'(x_0) = \frac{1}{2}\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{4} \]
\[ f''(x_0) = \frac{1}{4}\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{8} \]

Plug into \( p_0(x), p_1(x) \) and \( p_2(x) \), we can get:

\[ p_0(x) = -\frac{\sqrt{2}}{2} \]
\[ p_1(x) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\left(x + \frac{\pi}{2}\right) = -0.707 - 0.353(x + \frac{\pi}{2}) \]
\[ p_2(x) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\left(x + \frac{\pi}{2}\right) + \frac{\sqrt{2}}{16}\left(x + \frac{\pi}{2}\right)^2 = -0.707 - 0.353\left(x + \frac{\pi}{2}\right) + 0.0884\left(x + \frac{\pi}{2}\right)^2 \]

Sketch:
2.

**Constructing the second order Taylor Polynomial:**

\[ f(x) = p_2(x) + R_2(x) \]

Where

\[ p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \]

\[ f(x_0) = \ln(2) \]

\[ f'(x) = \frac{1}{x - 6}, \quad f'(x_0) = -\frac{1}{4} \]

\[ f''(x) = -\frac{1}{(x - 6)^2}, \quad f''(x_0) = -\frac{1}{16} \]

Plug into \( f(x) = p_2(x) + R_2(x) \)

\[ f(x) = \ln(2) - \frac{1}{4}(x - 2) - \frac{1}{32}(x - 2)^2 + R_2(x) = 0.693 - 0.25(x - 2) - 0.031(x - 2)^2 + R_2(x) \]

**Finding upper bound on the error:**
The remainder for a 2nd order Taylor is given by \( |R_2(x)| = \frac{|f^{(3)}(\xi)|}{3!}|x - x_0|^3 \), where \( x \) and \( \xi \) are an element of \([0,2]\) for \( x = 0 \) and \([2,4]\) for \( x = 4 \).

Consider the interval \([0,2]\) when \( x = 0 \):

\[
|R_2(x)| = \frac{|f^{(3)}(\xi)|}{3!}|x - x_0|^3 \leq \frac{\max}{3!}|x - x_0|^3
\]

Since \( |x - x_0|^3 \leq 8 \), that is, the term is max at \( x = 0 \)

Also \( |f^{(3)}(x)| = \left| \frac{2}{(x-6)^3} \right| \) increases as \( x \) increases, then \( |f^{(3)}(\xi)| \) is max at \( \xi = 2 \) on the interval of \([0,2]\)

In this case, the upper bound on the error at \( x = 0 \) is given by:

\[
|R_2(x)| \leq \frac{\max}{3!}|x - x_0|^3 = \frac{1}{24} = 0.0417
\]

Similarly, on the interval \([2,4]\) when \( x = 4 \):

\[
|R_2(x)| = \frac{|f^{(3)}(\xi)|}{3!}|x - x_0|^3 \leq \frac{\max}{3!}|x - x_0|^3
\]

Since \( |x - x_0|^3 \leq 8 \), that is, the term is max at \( x = 4 \)

Also \( |f^{(3)}(x)| = \left| \frac{2}{(x-6)^3} \right| \) increases as \( x \) increases, then \( |f^{(3)}(\xi)| \) is max at \( \xi = 4 \) on the interval of \([2,4]\)

In this case, the upper bound on the error at \( x = 0 \) is given by:

\[
|R_2(x)| \leq \frac{\max}{3!}|x - x_0|^3 = \frac{1}{3} = 0.33
\]

**Calculating the actual error:**

\[
actual\ error = |f(x) - p_2(x) |
\]

At \( x = 0 \):

\[
|f(0) - p_2(0)| = \ln(3) - \ln(2) - \frac{1}{2} + \frac{1}{8} = 0.03046
\]

At \( x = 4 \):

\[
|f(4) - p_2(4)| = \left| \ln(1) - \ln(2) + \frac{1}{2} + \frac{1}{8} \right| = 0.06815
\]
3.

As with problem 2, the remainder is given by

\[ R_n(x) = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \leq \frac{\max_{\xi \in [0,2]} |f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \]

Where \( x_0 = 1 \) over the interval \([0,2]\)

Since \( |x - x_0|^{n+1} \leq 1 \)

Need to find out the upper bound of \( \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \)

\[ f(x) = \ln(4 - x) \]
\[ f'(x) = \frac{1}{x - 4} \]
\[ f''(x) = -\frac{1}{(x - 4)^2} \]
\[ f^{(3)}(x) = \frac{2}{(x - 4)^3} \]
\[ f^{(4)}(x) = \frac{6}{(x - 4)^4} \]

It is apparent to see that the derivatives follow the form when \( n \geq 1 \):

\[ f^{(n)}(x) = \frac{(n-1)!}{(x - 4)^n} \]

Putting it into the Remainder formula, the form becomes:

\[ f^{(n+1)}(\xi) = \frac{n!}{(\xi - 4)^{n+1}} \]

The upper bound is given by (with \( x_0 = 1 \)):

\[ |R_n(x)| \leq \frac{\max_{\xi \in [1,2]} |f^{(n+1)}(\xi)|}{(n+1)!} |x - x_0|^{n+1} \leq \frac{\max_{\xi \in [1,2]} \frac{1}{(n+1)(\xi - 4)^{n+1}}}{(n+1)!} \]

We want to maximize the remainder over the separate intervals \([1,2]\) and \([0,1]\)

For \( x \in [1,2] \)

\[ |R_n(x)| \leq \frac{\max_{\xi \in [1,2]} \frac{1}{(n+1)(\xi - 4)^{n+1}}}{(n+1)!} \]

Since \( \frac{1}{(\xi - 4)^{n+1}} \) is maximized at \( \xi = 2 \) (denominator should be minimized to max the overall value)
\[ |R_n(x)| \leq \frac{1}{(n+1)2^{n+1}} \]  

(1)

Similarly, for \( x \in [0,1] \)

\[ |R_n(x)| \leq \max_{\xi \in [0,1]} \frac{1}{(n+1)(\xi - 4)^{n+1}} \times 1 \]

Since \( \frac{1}{(\xi - 4)^{n+1}} \) is maximized at \( \xi = 2 \) (denominator should be minimized to max the overall value)

\[ |R_n(x)| \leq \frac{1}{(n+1)3^{n+1}} \]  

(2)

Compare equations (1) and (2) we can get:

\[ |R_n(x)| \leq \frac{1}{(n+1)2^{n+1}} \quad \forall x \in [0,2] \]

Just need to plug in numbers \( n = 1, 2, 3, \ldots \) to satisfy \( |R_n(x)| \leq 0.0001 \)

When \( n = 9, |R_n(x)| = 0.00009765625 \leq 0.0001 \)

\[
\begin{align*}
f(x_0) &= \ln(3) = 1.09 \\
f'(x_0) &= \frac{1}{x_0 - 4} = -\frac{1}{3} \\
f''(x_0) &= -\frac{1}{(x_0 - 4)^2} = -\frac{1}{9} \\
f'''(x_0) &= -\frac{2}{27} \\
f^{(4)}(x_0) &= -\frac{6}{81} \\
f^{(5)}(x_0) &= -\frac{24}{243} \\
f^{(6)}(x_0) &= -\frac{120}{729} \\
f^{(7)}(x_0) &= -\frac{720}{2187} \\
f^{(8)}(x_0) &= -\frac{5040}{6561} \\
f^{(9)}(x_0) &= -\frac{40320}{19683}
\end{align*}
\]

The Taylor polynomials of order of 9 around \( x_0 = 1 \) is:
\[ p_9(x) = 1.09 - \frac{1}{3} (x - 1) - \frac{1}{18} (x - 1)^2 - \frac{1}{81} (x - 1)^3 - \frac{1}{324} (x - 1)^4 - \frac{1}{1215} (x - 1)^5 \\
- \frac{1}{4374} (x - 1)^6 - \frac{1}{15309} (x - 1)^7 - \frac{1}{52488} (x - 1)^8 - \frac{1}{177147} (x - 1)^9 \]

Actual error:

\[ |f(0) - p_9(0)| = 1.3 \times 10^{-6} \]
\[ |f(1) - p_9(1)| = 0 \]
\[ |f(2) - p_9(2)| = 2.433 \times 10^{-6} \]