Notation

$\mathbb{R}$ — real numbers
$f \in C(a, b)$ — f is a continuous function defined on the interval $[a, b]$
$f \in C^n(a, b)$ — f is n-times continuously differentiable on $[a, b]$
$f^{(n)}$ — the n-th derivative of f
$\overline{x_0 x_1}$ — the interval $[x_0, x_1]$ if $x_0 < x_1$, otherwise $[x_1, x_0]$

Taylor's theorem

Theorem (Thm. 1.1, p.3)

If $f \in C^{n+1}(a,b)$, then given any $x_0 \in [a,b]$,

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{p_n(x)} + R_n(x) \text{ for all } x \in [a,b],$$

where $R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,\mathrm{d}t$.
Moreover,

$$|R_n(x)| \leq \underbrace{\frac{|x - x_0|^{n+1}}{(n+1)!} \max_{\zeta \in \overline{x_0 x}} \left|f^{(n+1)}(\zeta)\right|}_{B_n(x)}.$$

Example

Let $f(x) = e^{x/2}$. Estimate the value of $f(-1)$ using the second-order Taylor expansion of f centered at $x_0 = 1$. Compute the error bound and the actual error.

We compute the derivatives and make the following table.

k	$f^{(k)}(x)$	$f^{(k)}(x_0)$
0	$e^{x/2}$	$e^{1/2}$
1	$\frac{1}{2}e^{x/2}$	$\frac{1}{2}e^{1/2}$
2	$\frac{1}{4}e^{x/2}$	$\frac{1}{4}e^{1/2}$
3	$\frac{1}{8}e^{x/2}$	$-$

Example (cont.)

Thus, we have

$$p_2(x) = e^{1/2}(1 + \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2),$$

which yields $p_2(-1) = \frac{1}{2}e^{1/2} \approx 0.8244$.
The actual value of f at -1 is $f(-1) = e^{-1/2} \approx 0.6065$. The actual error is therefore $|p_2(-1) - f(-1)| \approx 0.2178$.
The error bound is

$$B_2(-1) = \frac{2^3}{3!} \cdot \frac{1}{8}e^{1/2} \approx 0.2748.$$

We see that the error bound is greater than the actual error (as expected).