Corrected Trapezoid Rule I

It can be derived from the integral mean value theorem that

$$\int_a^b f(x)\,\mathrm{d}x \approx I_n^{\mathrm{TRAP}}(f) - \frac{h^2}{12}\underbrace{I_n^{\mathrm{MID}}(f'')}_{\int_a^b f''(x)\,\mathrm{d}x + \mathcal{O}(h^2)}$$

$$= I_n^{\mathrm{TRAP}}(f) - \frac{h^2}{12}(f'(b) - f'(a)) + \mathcal{O}(h^4).$$

This suggests that if we use $\mathcal{O}(h^2)$ approximations of $f'(a)$ and $f'(b)$, then we will have an $\mathcal{O}(h^4)$ method for integration.

Corrected Trapezoid Rule II

Using the $\mathcal{O}(h^2)$ asymmetric derivative approximation from homework, we have

$$\boxed{\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\approx I_n^{\mathrm{TRAP}}(f) - \frac{h^2}{12}\frac{1}{2h}\Big[3f(x_0) - 4f(x_1) + f(x_2) \\ &\qquad\qquad + f(x_{n-2}) - 4f(x_{n-1}) + 3f(x_n)\Big] \\ &= I_n^{\mathrm{TRAP}}(f) - \frac{h}{24}\Big[3f(x_0) - 4f(x_1) + f(x_2) \\ &\qquad\qquad + f(x_{n-2}) - 4f(x_{n-1}) + 3f(x_n)\Big].\end{aligned}}$$

This is an $\mathcal{O}(h^4)$ approximation of the integral.

Solving systems of linear equations I

Suppose we'd like to solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + 2x_3 &= 3 \\ 2x_1 + 2x_2 + 4x_3 &= 9. \end{aligned}$$

Putting it into matrix form, we have the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 2 & 4 & 9 \end{array}\right)$$

Solving systems of linear equations II

Performing Gaussian elimination, we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 0 & 2 & 3 \\ 2 & 2 & 4 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -1 & -1 \\ 0 & -2 & -2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right)$$

This puts the matrix into a upper triangular form, and the resulting equation can then be solved using back substitution. That is, $x_3 = \frac{2}{-1} = -2$, $x_2 = \frac{-1}{-2} - \frac{-1}{-2}x_3 = \frac{3}{2}$ and $x_1 = \frac{4}{1} - \frac{2x_2}{1} - \frac{3x_3}{1} = 7$.

Solving systems of linear equations III

Consider an $n \times n$ system in general.
Starting from the first column, gaussian elimination requires $n+1$ multiplications/divisions and n additions for each row (with $n-1$ rows in total). For the second column, it needs n multiplications/divisions and $n-1$ additions for each row (now with $n-2$ rows to go).
The total FLOPs required is on the order of $\mathcal{O}(n^3)$.

Sparse systems

Sparse matrices are ones whose entries are mostly zeros. A simple type of sparse matrices is a tridiagonal matrix.

$$\begin{pmatrix} \times & \times & & & & \\ \times & \times & \times & & & \\ & \times & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \times \\ & & & & \times & \times \end{pmatrix}$$