Integration

Given a function f and an interval $[a, b]$, we'd like to compute the integral $\int_a^b f(x)\,\mathrm{d}x$.

- Left endpoint rule
- Trapezoid rule
- Midpoint rule
- Corrected trapezoid rule
- ...

Left endpoint rule

Divide the interval $[a, b]$ into n pieces. Denote $x_0 = a, x_k = a + hk$.
Each piece has length $h = \frac{b-a}{n}$.
Then
$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{n-1} hf(x_k) = \underbrace{h\sum_{k=0}^{n-1} f(x_k)}_{I_n^{\mathrm{LE}}}.$$

Error analysis

In $[x_k, x_{k+1}]$, the error of this approximation is

$$\underbrace{\left|\int_{x_k}^{x_{k+1}} f(x)\,\mathrm{d}x - hf(x_k)\right|}_{\hat{e}_{n,k}^{\mathrm{LE}}} = \left|\int_{x_k}^{x_{k+1}} f(x) - f(x_k)\,\mathrm{d}x\right|$$

$$\leq \int_{x_k}^{x_{k+1}} |f(x) - f(x_k)|\,\mathrm{d}x$$

$$\leq \int_{x_k}^{x_{k+1}} B_0(x)\,\mathrm{d}x.$$

Assuming $|f'(x)| \leq K_1$ for all $x \in [a, b]$, we have
$\hat{e}_{n,k}^{\mathrm{LE}} \leq \int_{x_k}^{x_{k+1}} K_1(x - x_k)\,\mathrm{d}x = \frac{1}{2}K_1 h^2$.
Then

$$e_n^{\mathrm{LE}} \leq n \cdot \left(\frac{1}{2}K_1 h^2\right) = \boxed{\frac{1}{2}K_1(b-a)h} = \boxed{\frac{K_1(b-a)^2}{2n}}.$$

The error bound is only valid if $f \in C^1$.

Trapezoid method

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{n-1} \frac{1}{2}(f(x_k) + f(x_{k+1}))h$$
$$= \underbrace{h\left[\frac{1}{2}(f(a) + f(b)) + \sum_{k=1}^{n-1} f(x_k)\right]}_{I_n^{\mathrm{TRAP}}}.$$

Let $\ell(x)$ denote the line through $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$ and let $K_2 = \max_{x\in[a,b]} |f''(x)|$. Using Taylor polynomial based bounds (similar to how we analyzed linear interpolation), we have

$$\underbrace{\left|\int_{x_k}^{x_{k+1}} \ell(x) - f(x)\,\mathrm{d}x\right|}_{\hat{e}_{n,k}^{\mathrm{TRAP}}} \leq \int_{x_k}^{x_{k+1}} \frac{K_2}{2}(x - x_k)(x_{k+1} - x)\,\mathrm{d}x = \frac{K_2}{12}h^3.$$

Therefore, $\boxed{e_n^{\mathrm{TRAP}} \leq n \cdot \frac{K_2}{12}h^3 = \frac{K_2}{12}(b-a)h^2 = \frac{K_2}{12}(b-a)^3 n^{-2}}$.

Midpoint method

Let $\mu_k = \frac{x_k + x_{k+1}}{2}$ be the midpoint of $[x_k, x_{k+1}]$.

$$\int_a^b f(x)\,\mathrm{d}x \approx \underbrace{h \sum_{k=0}^{n-1} f(\mu_k)}_{I_n^{\mathrm{MID}}}.$$

Note that $hf(\mu_k) = \int_{x_k}^{x_{k+1}} f(\mu_k) + f'(\mu_k)(x - \mu_k)\,\mathrm{d}x$. Using Taylor polynomial based bounds (similar to the left endpoint analysis), we get

$$\left| \int_{x_k}^{x_{k+1}} f(x) - f(\mu_k)\,\mathrm{d}x \right| \leq \frac{K_2}{24} h^3.$$

Therefore, $\boxed{e_n^{\mathrm{MID}} \leq n \cdot \frac{K_2}{24} h^3 = \frac{K_2}{24}(b-a)h^2 = \frac{K_2}{24}(b-a)^3 n^{-2}}$.