Estimating the actual error for ODEs

Most ODEs don't have closed-form analytic solutions. Thus, we need a way to estimate "actual error" for ODE methods.

Recall from the previous lecture that the error at $t = T$ (i.e. $e_{n,n} = |x_n - \bar{x}(T)|$) is $\mathcal{O}(h)$.
If h is small, this order estimate essentially reduces to $e_{n,n} \approx Ch$ for some constant C. Therefore, $e_{10n,10n} \approx C \cdot \frac{h}{10} \ll e_{n,n}$. One may thus use the solution obtained using $10n$ steps as the "true" / "exact" solution when computing actual errors.

Linear interpolation

Given a function f and two data points $(x_0, f(x_0))$, $(x_1, f(x_1))$, the line through the two points is given by

$$\ell(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) = \frac{x_1 - x}{x_1 - x_0} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

In general, this line through the two data points may be arbitrarily far away from f between x_0 and x_1.

Error analysis

Assume $\max_{x\in[x_0,x_1]} |f''(x)| \leq C_2$. To simplify the analysis, suppose $f(x_0) = f(x_1) = 0$ and $x_0 = -x_1$. Let $h = x_1 - x_0$.
Because the second derivative is bounded, the worst case error occurs when the function is $f(x) = \pm\frac{C_2}{2}(x - \frac{h}{2})(x + \frac{h}{2})$.
The maximum error between f and ℓ is at $x = 0$; that is,

$$\boxed{\underbrace{\max_{x\in[x_0,x_1]} |\ell(x) - f(x)|}_{e^{\mathrm{la}}} \leq \frac{C_2}{2} \cdot \frac{h}{2} \cdot \frac{h}{2} = \frac{C_2}{8} h^2.}$$

Number of segments required to achieve a precision I

Given a function f on an interval $[a, b]$, suppose we'd like to obtain an approximation of f using linear interpolation using multiple segments, so that the approximation error is less than δ.

Subdivide $[a, b]$ into n equal segments. Then each segment has length $h = \frac{b-a}{n}$. The endpoints of the segments are $x_0 = a, x_1 = a + h, x_2 = a + 2h, \cdots, x_n = a + nh = b$.

From the error bound from previous slide, the maximum error over the k-th segment is bounded by $\frac{\max_{x \in [x_{k-1}, x_k]} |f''(x)|}{8} h^2$.

Number of segments required to achieve a precision II

To ensure the precision across $[a, b]$, we require each segment to have an error bound $\leq \delta$. That is, we require

$$\max_{k=1,\cdots,n} \frac{\max_{x\in[x_{k-1},x_k]} |f''(x)|}{8} h^2 \leq \delta.$$

Observe that

$$\max_{k=1,\cdots,n} \max_{x\in[x_{k-1},x_k]} |f''(x)| = \max_{x\in[a,b]} |f''(x)|.$$

Hence it suffices to require

$$\frac{\max_{x\in[a,b]} |f''(x)|}{8} h^2 = \boxed{\frac{\max_{x\in[a,b]} |f''(x)|}{8} \left(\frac{b-a}{n}\right)^2 \leq \delta}.$$