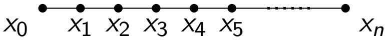
An example application: solving partial differential equations

Suppose we'd like to solve the partial differential equation $W_t(t, x) = F(x, W_x(t, x))$.[1]
Suppose the domain of the PDE has been discretized uniformly and we have $x_0, x_1, \cdots, x_n$.



Denote $h = x_1 - x_0 (= x_2 - x_1 = \cdots)$. Then

$$\begin{aligned} W(\delta, x_k) &\approx W(0, x_k) + \delta W_t(0, x_k) \\ &= W(0, x_k) + \delta F(x_k, W_x(0, x_k)) \\ &= W(0, x_k) + \frac{W(0, x_{k+1}) - W(0, x_{k-1})}{h}. \end{aligned}$$

[1] $W_t = \frac{\partial W}{\partial t}$, $W_x = \frac{\partial W}{\partial x}$.

An example application: solving partial differential equations

But this numerical scheme cannot be applied at the boundary $k = 0$ and $k = n$ (because we have no information about x_{-1} or x_{n+1}).
This motivates the asymmetric finite differences such as

$$W_x(0, x_0) \approx \frac{-\frac{3}{2}W(0, x_0) + 2W(0, x_1) - \frac{1}{2}W(0, x_2)}{h}.$$

Second derivative

Observe that

$$f'(x_0) \approx D_h(x_0) = \frac{f(x_0+h) - f(x_0)}{h}$$
$$f'(x_0-h) \approx D_h(x_0-h) = \frac{f(x_0) - f(x_0-h)}{h}.$$

Therefore,

$$f''(x_0) \approx \frac{f'(x_0) - f'(x_0-h)}{h} \approx \underbrace{\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}}_{D_h^2(x_0)}.$$

Error analysis

From the Taylor expansion of f at x_0, we have

$$
\begin{aligned}
f(x_0+h) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + R_3(x_0+h)\\
f(x_0-h) &= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{6}h^3 + R_3(x_0+h)\\
f(x_0) &= f(x_0)
\end{aligned}
$$

Combining them to form the numerator as in the last slide,

$$\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0)h^2 + R_3(x_0+h) + R_3(x_0-h).$$

Using the bound on R_3, we have

$$|D_h^2(x_0) - f''(x_0)| = \mathcal{O}(h^2).$$

Solving ODEs

initial value problem

Suppose we'd like to solve $\dot{x}(t) = f(t, x(t))$, $x(0) = x_0$ from $t = 0$ to $t = T$. First, we discretize the interval $[0, T]$ into n time steps.

$$0 = t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad \cdots \quad t_n = T$$

Let $h = \frac{T}{n}$ and let x_k denote our approximate solution at t_k. Since $\dot{x}(t) = f(t, x(t))$, we have

$$\dot{x}(0) \approx \frac{x_1 - x_0}{h} \quad \rightsquigarrow \quad x_1 = x_0 + f(t_0, x_0)h.$$

For subsequent steps, $\boxed{x_k = x_{k-1} + f(t_{k-1}, x_{k-1})h}$.

Error analysis I

Denote the (true/exact) solution of the ODE $\dot{x}(t) = f(t, x(t))$ as $\bar{x}$. Then

$$\begin{aligned}\bar{x}(t_1) &= x_0 + \dot{\bar{x}}(t_0)(t_1 - t_0) + R_1(t_1 - t_0) \\ &= x_0 + f(t_0, x_0)h + R_1(t_1 - t_0).\end{aligned}$$

Note that $t_1 - t_0 = h$. Comparing this with our approximation x_1, we see that

$$\begin{aligned}|\bar{x}(t_1) - x_1| = |R_1(h)| &\leq \frac{h^2}{2} \max_{r \in [0,h]} |\ddot{x}(r)| \\ &= \frac{h^2}{2} \underbrace{\max_{r \in [0,h]} \left| f_r(r, \bar{x}(r)) + f_x(r, \bar{x}(r)) f(r, \bar{x}(r)) \right|}_{K_2}.\end{aligned}$$

Suppose $\ddot{x}$ is bounded and denote the bound by K_2. Then $|\bar{x}(t_1) - x_1| \leq \frac{K_2}{2} h^2$.

Error analysis II

For the next step, we have

$$\begin{aligned}
\bar{x}(t_2) &= \bar{x}(t_1) + \dot{\bar{x}}(t_1)(t_2 - t_1) + R_1(t_2 - t_1) \\
&= \bar{x}(t_1) + f(t_1, x_1)h + R_1(t_2 - t_1) \\
x_2 &= x_1 + hf(t, x_2).
\end{aligned}$$

$$\begin{aligned}
|\bar{x}(t_2) - x_2| &= |\bar{x}(t_1) - x_1 + (\dot{\bar{x}}(t_1) - f(t_1, x_1))h + R_1(t_2 - t_1)| \\
&\leq |\bar{x}(t_1) - x_1| + |\underbrace{\dot{\bar{x}}(t_1)}_{f(t_1, \bar{x}(t_1))} - f(t_1, x_1)|h + |R_1(t_2 - t_1)|.
\end{aligned}$$

Error analysis III

Assume now

$$|f(t,x) - f(t,y)| \leq K_1|x-y| \text{ for all } x, y.$$

Then

$$|\bar{x}(t_2) - x_2| \leq \underbrace{|\bar{x}(t_1) - x_1|}_{\mathcal{O}(h^2)} + K_1|\bar{x}(t_1) - x_1|h + \underbrace{|R_1(t_2 - t_1)|}_{\mathcal{O}(h^2)}.$$

Using the bound $|\bar{x}(t_1) - x_1| \leq \frac{K_2}{2}h^2$. We see that $|\bar{x}(t_2) - x_2| = \mathcal{O}(h^2)$. Repeating this process, we see that for a fixed n, $e_{n,k} = |\bar{x}(t_k) - x_k| = \mathcal{O}(h^2)$ for each $k = 1, \cdots, n$. Since $h = \frac{T}{n}$, as we increase n and thereby reduce h, we must take more steps. Hence the error at the final time T converges to 0 on the order of h (i.e. $e_{n,n} = \mathcal{O}(h)$.)