Example: Asymptotic order

Let $f(x) = \cos(x) - 1$. Find the order of $f(x)$ as $x \downarrow 0$.

Write $f(x) = p_2(x) + R_2(x)$ with $x_0 = 0$. $p_2(x)$ in this case is given by

$$p_2(x) = \frac{-1}{2!}(x-0)^2 = -\frac{x^2}{2}.$$

For R_2, we have

$$|R_2(x)| \leq B_2(x) = \frac{|x|^3}{3!} \max_{\zeta \in [0,x]} |f'''(\zeta)| \leq \frac{|x|^3}{6}.$$

Hence $|f(x)| \leq |p_2(x)| + |R_2(x)| \leq \dfrac{x^2}{2} + \dfrac{x^3}{6}$ for all $x > 0$.
Using the method from the previous lecture, we have

$$|f(x)| \leq \frac{x^2}{2} + \frac{x^2}{6} = \frac{1}{3}x^2 \quad \text{for all } x \in (0,1).$$

We see that $f(x) = \mathcal{O}(x^2)$.

Catastrophic subtraction I

Consider a 5-digit machine. Suppose we need to evaluate (the first-order finite difference approximation of the derivative of f)

$$f'(x) \approx \frac{f(x+\delta) - f(x)}{\delta} \quad \text{where } f(x) = x^3$$

with some very small δ.
If we evaluated it with $x = 6$ and $\delta = 10^{-5}$, then

$$f(x) = 216, \quad f(x+\delta) = 216.0010800018.$$

But since the machine only keeps track of five digits, $f(x+\delta)$ would be treated as 216.00, and $\frac{f(x+\delta)-f(x)}{\delta}$ would be 0 on this machine.

Catastrophic subtraction II

If we evaluated it instead with $\delta = 10^{-4}$, then

$$f(x) = 216, \quad f(x+\delta) = 216.01080018.$$

But since the machine only keeps track of five digits, $f(x+\delta)$ would be treated as 216.01, and $\frac{f(x+\delta)-f(x)}{\delta}$ would be $\frac{0.01}{10^{-4}} = 100$ on this machine.
Compare this to the actual derivative $f'(6) = 108$.

Floating-point numbers

Single-precision number (32 bits): 1 sign bit + 8 bits for the exponent + 23 $_{(+1)}$ bits for the significand. The number is represented as

$$\underbrace{\pm}_{\text{sign}} \underbrace{(110110\cdots10111)_{\text{bin}}}_{\text{significand (24 bits)}} \times 2^{\overbrace{}^{}\underbrace{(00110101)_{\text{bin}}}_{\text{exponent (8 bits)}}} .$$

Double-precision number (64 bits): 1 sign bit + 11 bits for the exponent + 52 $_{(+1)}$ bits for the significand. The number is represented similarly.

The $_{(+1)}$ is because the significand (binary fraction) has a implicit integer part of 1 which is not stored, except for special values (denormals etc).

Horner's method I

Consider $f(x) = e^x$ and its Taylor expansion around $x = 0$, given by

$$\begin{aligned} f(x) &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4. \end{aligned}$$

Then we may rewrite the polynomial as

$$\begin{aligned} f(x) &= a_1 + x(a_2 + a_3 x + a_4 x^2 + a_5 x^3) \\ &= a_1 + x(a_2 + x(a_3 + a_4 x + a_5 x^2)) \\ &= a_1 + x(a_2 + x(a_3 + x(a_4 + a_5 x))). \end{aligned}$$

For a degree n polynomial, this requires n additions and n multiplications.

Horner's method II

Pseudocode for Horner's method

GIVEN $\mathtt{x}$, a
$\mathtt{PTEM} = a_5$
FOR $\mathtt{K}$ FROM 4 DOWN TO 1 BY -1's
  $\mathtt{PTEM} = \mathtt{PTEM} \cdot \mathtt{x} + a_{\mathtt{K}}$
END
$\mathtt{POUT} = \mathtt{PTEM}.$