Limit notation

Example (Continuous function) $\lim_{x\to 0} e^{x/2} = e^{0/2} = 1$.

Example (Heaviside function h)

$$h = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0. \end{cases}$$

$$\lim_{x \downarrow 0} h(x) = 1, \qquad \lim_{x \uparrow 0} h(x) = -1.$$

Example (Asymptotic behavior) $\lim_{x\to\infty} 2^{-x} = 0$.

Example of asymptotic order

Let $f(x) = \frac{10^{-3}}{x}$, $g(x) = 2^{-x}$.

x	$f(x)$	$g(x)$
1	10^{-3}	0.5
10	10^{-4}	$2^{-10} \approx 10^{-3}$
100	10^{-5}	$2^{-100} \approx 10^{-30}$

Although $f(x)$ was for small x, $g(x)$ was much smaller as x gets large.

Asymptotic order ($x \downarrow 0$)

We say $f(x) = \mathcal{O}(x^n)$ (f is order of x^n) as $x \downarrow 0$ if there exists some finite C and some $\delta > 0$ such that

$$|f(x)| \leq Cx^n \quad \text{for all } x \in (0, \delta);$$

equivalently, $-Cx^n \leq f(x) \leq Cx^n$ for all $x \in (0, \delta)$.

Example

Find the asymptotic order of $f(x) = 5x^2 - 20x^4$ as $x \downarrow 0$.
We have

$$|f(x)| = |5x^2 - 20x^4| \leq |5x^2| + |20x^4| = 5x^2 + 20x^4.$$

Recall that $x^4 \leq x^2$ if $x \in [0,1]$. Therefore,

$$|f(x)| \leq 5x^2 + 20x^2 = 25x^2 \quad \text{for all } x \in [0,1].$$

By choosing $C = 25$ and $\delta = 1$, we see that $f(x) = \mathcal{O}(x^2)$.

$\boxed{\text{It is implicit that we look for the “tightest” order (that is, largest } n\text{).}}$

Asymptotic order ($x \to \infty$)

We say $f(x) = \mathcal{O}(x^{-k})$ as $x \to \infty$ if
there exists some finite C and D such that

$$|f(x)| \leq Cx^{-k} \quad \text{for all } x > D;$$

equivalently, $-Cx^{-k} \leq f(x) \leq Cx^{-k}$ for all $x > D$.

Example

Find the asymptotic order of $f(x) = 5x^{-2} - 20x^{-4}$ as $x \to \infty$.
We have

$$|f(x)| = |5x^{-2} - 20x^{-4}| \leq |5x^{-2}| + |20x^{-4}| = 5x^{-2} + 20x^{-4}.$$

Recall that $x^{-4} \leq x^{-2}$ if $x \geq 1$. Therefore,

$$|f(x)| \leq 5x^{-2} + 20x^{-2} = 25x^{-2} \quad \text{for all } x \in [0, 1].$$

By choosing $C = 25$ and $D = 1$, we see that $f(x) = \mathcal{O}(x^{-2})$.

Example 2

Find the asymptotic order of $f(x) = \frac{1}{x^2+4x^5}$ as $x \to \infty$.
We have

$$|f(x)| = \frac{1}{x^2 + 4x^5} \leq \frac{1}{4x^5} \quad \text{for all } x > 0.$$

By choosing $C = \frac{1}{4}$, $D = 0$ (or any positive real number), we see that $f(x) = \mathcal{O}(x^{-5})$.

Example 3

Find the asymptotic order of $f(x) = \max\{\frac{1}{x^2}, \frac{5}{x^5}\}$ as $x \to \infty$.
We have

$$|f(x)| = \max\{\frac{1}{x^2}, \frac{5}{x^5}\} = \frac{1}{x^2} \quad \text{for all } x \text{ such that } \frac{5}{x^5} \leq \frac{1}{x^2}.$$

Note that $\frac{5}{x^5} \leq \frac{1}{x^2}$ holds when $x \geq \sqrt[3]{5}$. By choosing $C = 1$, $D = \sqrt[3]{5}$, we see that $f(x) = \mathcal{O}(x^{-2})$.