

Example: Minimum order Taylor polynomial to achieve a given precision

Let $f(x) = e^{x/2}$. Find the smallest n such that the approximation error for the n -th order Taylor polynomial with $x_0 = 1$ is guaranteed to be less than 0.1 over the interval $[-1, 3]$.

We want $|f(x) - p_n(x)| = |R_n(x)| \leq 0.1$ for all $x \in [-1, 3]$. Since $|R_n(x)| \leq |B_n(x)|$, we ensure that $|B_n(x)| \leq 0.1$ for all $x \in [-1, 3]$. Note that the worst (largest) B_n always occurs at an endpoint of the interval.

$$\begin{aligned} B_n(-1) &= \max_{\zeta \in [-1, 1]} |f^{(n+1)}(\zeta)| \frac{|-1 - 1|^{n+1}}{(n+1)!} \\ &= \frac{1}{2^{n+1}} e^{1/2} \frac{2^{n+1}}{(n+1)!} \\ &= \frac{e^{1/2}}{(n+1)!}. \end{aligned}$$

Example (cont.)

Similarly at $x = 3$, we have

$$\begin{aligned} B_n(3) &= \max_{\zeta \in [1,3]} |f^{(n+1)}(\zeta)| \frac{|3-1|^{n+1}}{(n+1)!} \\ &= \frac{1}{2^{n+1}} e^{3/2} \frac{2^{n+1}}{(n+1)!} \\ &= \frac{e^{3/2}}{(n+1)!}. \end{aligned}$$

Example (cont.)

Again, we tabulate the error bound for each n as follows.

k	$B_n(-1)$	$B_n(3)$
0	1.6487	4.4817
1	0.8244	2.2408
2	0.2748	0.7469
3	0.0687	0.1867
4	0.0137	0.0373

Hence the minimum order n to achieve a precision of 0.1 over $[-1, 3]$ is 4.

Floating-point operations

The number of floating-point operations (FLOPs) roughly corresponds with the computational cost.

The two common floating-point operations are addition-subtraction “A” and multiplication-division “M”.

In many cases, it's possible to optimize away a few FLOPs when evaluating polynomials.

Counting FLOPs

For example, to evaluate $p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ — a terrible implementation may actually compute

$$x - \frac{x \cdot x \cdot x}{3 \times 2 \times 1} + \frac{x \cdot x \cdot x \cdot x \cdot x}{5 \times 4 \times 3 \times 2 \times 1} \quad (2A + 14M),$$

whereas a better implementation may precompute the constants and store x^2 and x^3 as intermediate variables and reuse them.

We'll introduce a general method for evaluating polynomials efficiently next time.