Runge Kutta methods I

A second order Runge-Kutta method is one that is in the form

$$\begin{aligned} x_{k+1} &= x_k + \beta h \overbrace{f(t_k, x_k)}^{f_1} + \gamma h f(t_k + \alpha h, x_k + \alpha h f(t_k, x_k)) \\ &= x_k + [\beta + \gamma] f_1 h + \gamma\alpha [f_t + f_x f_1] h + \mathcal{O}(h^3). \end{aligned}$$

$$\bar{x}(t_{k+1}) = \bar{x}(t_k) + \dot{\bar{x}}(t_k) h + \frac{\ddot{\bar{x}}(t_k)}{2} h^2 + \mathcal{O}(h^3).$$

Thus we choose $\beta + \gamma = 1$ and $\alpha\gamma = \frac{1}{2}$. In line with the observation last time, this method is $\mathcal{O}(h^2)$ (in terms of global error).

Runge Kutta methods II

When implementing the algorihtm, given t_k, x_k, it is customary to write

$$\begin{aligned} K_1 &= hf(t_k, x_k) \\ K_2 &= hf(t_k + \alpha h, x_k + \alpha K_1) \\ x_{k+1} &= x_k + \beta K_1 + \gamma K_2. \end{aligned}$$

This approach is easily expandable to accommodate higher order RK methods.

A common choice for the parameters is $\alpha = 1, \gamma = \frac{1}{2}, \beta = \frac{1}{2}$.

RK45

The fourth-order Runge Kutta method (RK45), is given by

$$
\begin{aligned}
K_1 &= hf(t_k, x_k) \\
K_2 &= hf(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_1) \\
K_3 &= hf(t_k + \frac{h}{2}, x_k + \frac{1}{2}K_2) \\
K_4 &= hf(t_k + h, x_k + K_3) \\
x_{k+1} &= x_k + \frac{1}{6}K_1 + \frac{1}{3}K_2 + \frac{1}{3}K_3 + \frac{1}{6}K_4.
\end{aligned}
$$

This method has $\mathcal{O}(h^5)$ local error and $\mathcal{O}(h^4)$ global error, and requires 4 function evaluations per step.