Error analysis – Fixed point method

The error at k-th step is given by $e_k = |x_k - \bar{x}|$. Then

$$e_{k+1} = |x_{k+1} - \bar{x}| = |g(x_k) - g(\bar{x})| \leq K|x_k - \bar{x}| = Ke_k.$$

By induction, we have $\boxed{e_n \leq K^n e_0}$.
On the other hand,

$$\begin{aligned} e_0 = |x_0 - \bar{x}| \leq |x_0 - x_1| + |x_1 - \bar{x}| = |x_0 - x_1| + |g(x_0) - g(\bar{x})| \\ \leq |x_0 - x_1| + K \underbrace{|x_0 - \bar{x}|}_{e_0}. \end{aligned}$$

Therefore, $e_0 \leq |x_0 - x_1| + Ke_0$ and this can be used to obtain an estimate for e_0.

Fixed point method: n-D case

The fixed point method still works for equations of n unknowns (with the same convergence condition and error estimates in the scalar case) provided that there exists some $K < 1$ such that

$$\|g(\vec{x}) - g(\vec{y})\| \leq K\|x - y\| \text{ for all } x, y.$$

ODEs revisited

Consider

$$\dot{x} = f(t, x), \quad x(0) = x_0.$$

Let h denote the step size. From our error analysis for Euler's method, we had

$$e_{k+1} \leq (1 + Kh)e_k + \frac{M}{2}h^2,$$

where K and M are constants related to f and $|\ddot{x}|$.
With some additional work, it is possible to show

$$e_n \leq \frac{e^{KT}}{2K} Mh = \mathcal{O}(h).$$

Definition: The *local (residual) error* is e_{k+1} given $e_k = 0$. The *global (truncation) error* is $|x_k - \bar{x}(t_k)|$, given $e_0 = 0$.

The local error for Euler's method is $\mathcal{O}(h^2)$, but the global error is $\mathcal{O}(h)$. This loss of order from local to global error also holds for higher order Runge-Kutta methods.