Error analysis – Newton's method I

The Taylor expansion of f at x_k is

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \mathcal{O}(|x - x_k|^3).$$

Therefore,

$$0 = f(\bar{x}) \approx f(x_k) + f'(x_k)(\bar{x} - x_k) + \frac{f''(x_k)}{2}(\bar{x} - x_k)^2$$

Solving for $\bar{x}$, we have

$$\bar{x} \approx \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} - \frac{f''(x_k)}{2f'(x_k)}(x - x_k)^2$$

Error analysis – Newton's method II

Therefore,

$$\underbrace{|\bar{x} - x_{k+1}|}_{e_{k+1}} \lesssim \left| \frac{f''(x_k)}{2f'(x_k)} \right| \underbrace{|\bar{x} - x_k|}_{e_k}^2.$$

This shows quadratic convergence when the the guess x_k is close enough to $\bar{x}$.

Secant method

Using the derivative approximation

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

in the recurrence formula in Newton's method, we will instead have

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{\frac{f(x_k)-f(x_{k-1})}{x_k - x_{k-1}}}}.$$

This is the secant method, which only requires one function evaluation per step.

Example

Solve $f(x) = x^3 + 2x - 3 = 0$ using secant method with $x_0 = 3$, $x_1 = 2$. $(\bar{x} = 1)$

Using the formula in the previous slide, we have

$$x_2 = x_1 - \frac{f(x_1)}{\frac{f(x_1) - f(x_0)}{x_1 - x_0}} = 2 - \frac{9}{\frac{30-9}{1}} = \frac{11}{7} \approx 1.57143.$$

$$x_3 = x_2 - \frac{f(x_2)}{\frac{f(x_2) - f(x_1)}{x_2 - x_1}} \approx 1.22496.$$

Fixed point method

A fixed point of a function g is a point $\bar{x}$ such that $g(\bar{x}) = x$. The fixed point method find a fixed point rather than a root.

The fixed point iteration is given by

$$\boxed{x_{k+1} = g(x_k)}.$$

Solve $x^3 = x + 6$ using the fixed point method using $x_0 = 21$. (The exact root $\bar{x} = 2$.)
We arrange it into a fixed point form $x = (x+6)^{1/3}$. Using the iteration formula, we have

$$x_1 = (21+6)^{1/3} = 3$$
$$x_2 = (3+6)^{1/3} \approx 2.080$$
$$x_3 = (2.080+6)^{1/3} \approx 2.007.$$

If we instead arranged it into another form $x = x^3 - 6$, then the iteration would not have converged even if we started at $x_0 = 2.1$.

Convergence

Suppose there exists $K < 1$ such that $|g(x) - g(y)| \leq K|x - y|$ for all x, y. (In particular, this holds if $|g'(x)| \leq K < 1$ for all x.[1])

Then $|x_{k+1} - x_k| = |g(x_k) - g(x_{k-1})| \leq K|x_k - x_{k+1}|$ and the sequence x_k converges to some limit $\hat{x}$.
Therefore,

$$g(\hat{x}) = \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} x_{k+1} = \hat{x}.$$

This shows that $\hat{x}$ is a fixed point.

If $\hat{x}, \bar{x}$ are two fixed points, then

$$|\hat{x} - \bar{x}| = |g(\hat{x}) - g(\bar{x})| \leq K|\hat{x} - \bar{x}|.$$

Therefore we must have $|\hat{x} - \bar{x}| = 0$, which implies $\hat{x} = \bar{x}$.

Conclusion: for such a g, the fixed point method converges and there is only one solution to $x = g(x)$.

[1] This may fail to hold even if $|g'(x)| < 1$ for all x.