Root finding methods

A *root* or a *zero* of a function $f(x)$ is a point $\bar{x}$ such that $f(\bar{x}) = 0$. Root finding is equivalent to solving an equation.

- Bisection method
- Newton's method
- Secant method
- ...

Bisection method I

Suppose we'd like to find a root of $f(x)$, and we know there are a_0, b_0 with $a_0 < b_0$ such that $f(a_0)$ and $f(b_0)$ have opposite signs. If f is continuous, then there is a root of f in (a_0, b_0).

¶ To find this root, let $c_1 = \frac{1}{2}(a_0 + b_0)$.
If $f(c_1) = 0$, c_0 is a root and we are done.
If $f(c_1)$ and $f(a_0)$ have opposite signs, let $a_1 = a_0$, $b_1 = c_1$ and repeat from ¶ (now with a_1, b_1 replacing a_0, b_0).
Otherwise, $f(c_1)$ and $f(b_0)$ have opposite signs; we let $a_1 = c_1$, $b_1 = b_0$ and repeat from ¶.

Bisection method II

Each step of iteration requires one function evaluation.

At each step, there is a root in the interval (a_k, b_k). Without further information about f, we take $c_{k+1} = \frac{1}{2}(a_k + b_k)$ (the midpoint) as the approximation of root. This choice minimizes the worst case error.
The error at n-th step is therefore $e_n \leq \frac{b_0 - a_0}{2^n}$.[1] To get within ϵ of a root in (a_0, b_0), we'd require

$$\frac{b_0 - a_0}{2^n} \leq \epsilon.$$

Then iterating n steps yields an approximant of accuracy ϵ.

Bisection method is very robust — the error is reduced every step.

[1] Be careful how you count steps (0-based vs. 1-based).

Newton's method

Given an initial guess x_0 of a root, Newton's method approximates/replaces the function f by its tangent $y(x) = f(x_0) + f'(x_0)(x - x_0)$. This yields the next guess $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.
In general, the recurrence formula is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Each iteration requires one function evaluation and one derivative evaluation.

Example

Solve $x^3 + 3x = 4$ with initial guess 2.

We now that the true root is $\bar{x} = 1$.
Rearrange into root finding form: $f(x) = x^3 + 3x - 4$. To apply Newton's method, we find $f'(x) = 3x^2 + 3$.
Starting from $x_0 = 2$.

k	x_k	$e_k = \lvert x_k - \bar{x} \rvert$
0	2	1
1	1.333333	$\frac{1}{3}$
2	1.048889	4.9×10^{-2}
3	1.001175	1.2×10^{-3}
4	1.000001	6.9×10^{-7}

The convergence is very fast. But Newton's method is not robust and may fail to converge depending on the initial guess.