Sensor Attack Avoidance and Asset Protection: Risk-Averse and Linear-Quadratic Multi-Agent Differential Game Approach

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Technical Approach & Contributions
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    ▪ Determinants of Performance Information Value
    ▪ Attributes of Performance Information Value
    ▪ Dynamics of Performance Information
  • Decision Making under Uncertainty
    ▪ A Risk-Value Model
    ▪ Measures of Performance Risk
    ▪ Preference Model for Decision Making
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Conclusions
Introduction: What is this AFRL?
Introduction:
So what is Space Vehicles Directorate then?

Flight Experiments
- C/NOFS
- DSX
- XSS-11

Research
- Power & Photovoltaics
- Detectors & FPA’s
- Radiation Hard Electronics
- Structures
- Dynamics & Controls
- Ionosphere & effects
- Space Environment

Tech Development
- Laser Communications
- Remote Sensing
- Precision Navigation

http://www.vs.afrl.af.mil/

Motivations: Autonomous Reporting System

**Operational Capability**
- System design guidelines on *whether to actively sense, what* sensing actions to take, and *how much gain to expect in terms of* enhancing the situation awareness
- Network scalability wherein each space sensor compete only for its own performance without attempting on others’ behalf
- Autonomous event reporting via multi-agent frameworks; adversarial strategies and models

**Technology Challenges**
- Detection, tracking, and identification of large numbers of intelligent objects with all intervention/communication/detection latencies
- Adaptive coordination of sensor dwells and motions, in accordance with power, weather conditions, aerodynamic, earth blockages, obscurations and other constraints
- Situation awareness metrics guiding sensor and network resources allocation to data acquisition, interpretation & decision making

**Far Term Experiment/Demo**
- Persistent awareness demonstration of dynamic spatial-temporal regions
- Opportune and autonomous data acquisition using compressive sensing and active sensing
- Prototype of accurate tracking of maneuvers (e.g., changes in altitudes, inclinations, and phases) and time critical objects in preference with neutral and protected objects
- Demonstration of autonomous multi-level sequential surveillance strategy & visualization
Motivations: Autonomous Reporting System

Assumptions and Boundaries

• Time critical object capability: kinematics and dynamics of platforms; traverse speeds and times; levels of intelligence and awareness
• Capabilities of of neutral objects and assets being protected (e.g., stationary, orbit transfer)
• Effects based sensor modeling: reaction time; probability of detection; probability of acquisition, signature values; effective radius; max SBV and Radar ranges & resolutions

Performance Metrics

• Responsive Measurants: Timeliness access, maneuver time, detection time, gap time, ignorance cost, probability of occurrence
• Operational Measurants: Availability for high priority task; coverage vs. redundancy, usefulness, confidence, and probability of correct classification
• Affordable Measurants: Observation efficiency, collection robustness, mean time between sorties, endurance, and productivity

Mission CONOPS

• Uncorrelated objects in the mist of high density environments; Objects’ tactics favor use of protection from uncertain vicinity
• All of altitude, inclination, and phase changes are important for catalog updates for abnormal maneuvers \rightarrow affording opportune surveillance
• Maintenance of transient behaviors of time critical objects against all facets of numerous short transitions and followed by deception
Motivations: Mission Impacts and Benefits

• Coordination for Effects and Tactical Advantage
  - Cooperative Sensor Attack Avoidance
  - High-Value Asset Protection

• Training, Exercise, and Supportability
  - Early Missile Warning
  - On-Orbit Maneuvers
  - Anomaly Resolution and Prediction

• Multi-Agent Differential Game Theory \( \rightarrow \) A Natural Framework
  - Mathematical models for offense and defense engagement
  - Approaches for autonomous offensive and defensive tactics
  - Performance robustness beyond statistical averaging

Enabling Capabilities: 1) Know How; 2) Know-to-Cooperate; and 3) Need-to-Cooperate
Adversarial Hierarchies: Coalition-Conscious Interaction Dynamics

Let
\[ x^X \triangleq \begin{bmatrix} (x_1^X)^T, \ldots, (x_{m^x}^X)^T \end{bmatrix}^T \quad A^X \triangleq \text{diag} \left( A_1^X, \ldots, A_{m^x}^X \right) \quad G^X \triangleq \text{diag} \left( G_1^X, \ldots, G_{m^x}^X \right) \]
\[ u^X \triangleq \begin{bmatrix} (u_1^X)^T, \ldots, (u_{m^x}^X)^T \end{bmatrix}^T \quad B_u^X \triangleq \text{diag} \left( B_{u_1}^X, \ldots, B_{u_{m^x}}^X \right) \quad X = A, S, T \]

\[ dx^X (t) = \left( A^X x^X (t) + B_u^X u^X (t) \right) dt + G^X d\omega^X (t) \quad x^X (t_0) = x_0^X \quad t \in [t_0, t_f] \]

Team vs. team confrontations

\[ dx(t) = \left( Ax(t) + B^A u^A (t) + B^S u^S (t) + B^T u^T (t) \right) dt + G(t) d\omega (t) \quad x(t_0) = x_0 \quad t \in [t_0, t_f] \]

where
\[ x \triangleq \begin{bmatrix} (x^A)^T, (x^S)^T, (x^T)^T \end{bmatrix}^T \quad B^\Delta \triangleq \begin{bmatrix} (B_u^A)^T, 0, 0 \end{bmatrix}^T \quad B^T \triangleq \begin{bmatrix} 0, 0, (B_u^T)^T \end{bmatrix}^T \quad G \triangleq \text{diag} \left( G^A, G^S, G^T \right) \]
\[ A \triangleq \text{diag} \left( A^A, A^S, A^T \right) \quad B^S \triangleq \begin{bmatrix} 0, (B_u^S)^T, 0 \end{bmatrix}^T \quad w \triangleq \begin{bmatrix} (w^A)^T, (w^S)^T, (w^T)^T \end{bmatrix}^T \]

Information Flow Structure & Levels of Cooperation for Autonomous Teams
Adversarial Hierarchies: Encountered Environments with Basic Elements Known

Uncertain environments consisted of un-modeled nonlinearities & disturbances with mixed random realizations defined on \( \left( \Omega_i^X, F_i^X, \left\{ F^X_{i,t_f \geq t_0} \right\}, P_i^X \right) \)

\[
w_i^X(t) \triangleq w_i^X(t, \omega_i^X): [t_0, t_f] \times \Omega_i^X \mapsto \mathbb{R}^{p_i^X}; \quad X = A, S, T \text{ and } i = 1, \ldots, m_X
\]

\[
E \left\{ w_i^X(\tau) - w_i^X(\sigma) \left[ w_i^X(\tau) - w_i^X(\sigma) \right]^T \right\} = W_i^X |\tau - \sigma|; \quad \forall \tau, \sigma \in [t_0, t_f]
\]

Thus, aggregate uncertain environments with a-priori characteristics become

\[
w(t) \triangleq w(t, \omega): [t_0, t_f] \times \Omega \mapsto \mathbb{R}^p
\]

\[
E \left\{ w(\tau) - w(\sigma) \left[ w(\tau) - w(\sigma) \right]^T \right\} = W |\tau - \sigma|; \quad \forall \tau, \sigma \in [t_0, t_f]
\]

where

\[
W \triangleq \text{diag} \left( W_1^A, \ldots, W_{m_A}^A, W_1^S, \ldots, W_{m_S}^S, W_1^T, \ldots, W_{m_T}^T \right)
\]

Risk and Uncertainty Being Injected by Additive Stochastic Disturbances
Adversarial Hierarchies: Respective Tradeoff Strategies and Preferences

\[ u_i^X \in U_i^X \subseteq L^2_{F_i} \left( t_0, t_f; \mathbb{R}^{n_i} \right) \quad \text{and} \quad U^X \equiv \bigotimes_{i=1}^{m_X} U_i^X \]

Associated with an admissible 4-tuple \( (x(\cdot); u^A(\cdot); u^S(\cdot); u^T(\cdot)) \)

\[
J^r_i \left( x^0; u^A(\cdot), u^S(\cdot), u^T(\cdot) \right) = w_{ir}^S \left\| P \left( x_i^S (t_f) \right) - P \left( x_i^T (t_f) \right) \right\|^2 - \sum_{j=1}^{m_A} w_{ij}^A \left\| P \left( x_j^A (t_f) \right) - P \left( x_i^S (t_f) \right) \right\|^2
\]

\[
+ \int_{t_0}^{t_f} \left[ \sum_{j=1}^{m_S} \left( u_j^S \right)^T (\tau) R_{ij}^{SS} u_j^S (\tau) - \sum_{j=1}^{m_A} \left( u_j^A \right)^T (\tau) R_{ij}^{SA} u_j^A (\tau) - \sum_{j=1}^{m_T} \left( u_j^T \right)^T (\tau) R_{ij}^{ST} u_j^T (\tau) \right] d\tau
\]

\[
+ \int_{t_0}^{t_f} \left[ \sum_{j=1}^{m_A} \bar{w}_{ij}^A \left\| P \left( x_j^A (\tau) \right) - P \left( x_i^T (\tau) \right) \right\|^2 - \sum_{j=1}^{m_T} \bar{w}_{ij}^T \left\| P \left( x_j^T (\tau) \right) - P \left( x_i^S (\tau) \right) \right\|^2 \right] d\tau ; \quad r = 1, \ldots, m_r \quad i = 1, \ldots, m_S
\]

\( P : \mathbb{R}^{n_i^X} \mapsto N \) with \( P \left( x_i^X \right) = \left[ x_{i1}^X, \ldots, x_{in_0}^X \right]^T \in N \subseteq \mathbb{R}^{n_0} \)

\( R_{ij}^{SA} \in \mathbb{R}^{m_i \times m_j}, R_{ij}^{SS} \in \mathbb{R}^{m_i \times m_j}, R_{ij}^{ST} \in \mathbb{R}^{m_j \times m_j}, \bar{w}_{ir}^S, \bar{w}_{ij}^A, w_{ir}^S, w_{ij}^A \) -- weighting scalars

Implementation of Stochastic Tracking Preferences
Adversarial Hierarchies:
Simultaneous One-on-One Engagements

\[ J_i \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) = \begin{bmatrix} J_i^1 \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) \\ J_i^2 \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) \\ \vdots \\ J_i^{m_i} \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) \end{bmatrix}; \quad i = 1, \ldots, m_S \]

For a given set of \( \alpha^S_r \geq 0 \) and \( \sum_{r=1}^{m_r} \alpha^S_r = 1 \), the set of Pareto measures is given by

\[ J_i \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) = \sum_{r=1}^{m_r} \alpha^S_r J_i^r \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right); \quad i = 1, \ldots, m_S \]

\[ J_i \left( x_0, u^A (\cdot), u^S (\cdot), u^T (\cdot) \right) = x^T (t_f) Q^{SS}_{ij} x(t_f) + \int_{t_0}^{t_f} x^T (\tau) Q^{SS}_{i} x(\tau) d\tau \]

\[ + \int_{t_0}^{t_f} \left[ \sum_{j=1}^{m_S} (u_j^S)^T (\tau) R^{SS}_{ij} u_j^S (\tau) - \sum_{j=1}^{m_A} (u_j^A)^T (\tau) R^{SA}_{ij} u_j^A (\tau) - \sum_{j=1}^{m_T} (u_j^T)^T (\tau) R^{ST}_{ij} u_j^T (\tau) \right] d\tau \]
Adversarial Hierarchies: Pareto Coordination for Structuring Coalition Interactions

Consider a **Pareto team** defined by

1. A finite set of team members, \( \overline{S} \triangleq \{1, 2, \ldots, S\} \subseteq T \)
2. A collection of decision sets indexed on \( \overline{S} \), \( \{U_j\}_{j \in \overline{S}} \)
3. A payoff functional \( J : \times_{j=1}^{m_A} U^A_j \times \times_{j=1}^{m_T} U^T_j \rightarrow \mathbb{R}_+ \)

Finding **efficient and collective decisions** such that

\[
U^A \triangleq \times_{j=1}^{m_A} U^A_j; \quad U^S \triangleq \times_{i=1}^{S^S} U^S_i; \quad U^T \triangleq \times_{j=1}^{m_T} U^T_j
\]

\[
\arg\min_{u^A \in U^A, u^S \in U^S, u^T \in U^T} \left\{ J_1 \left( x_0; u^A, u^S, u^T \right), \ldots, J_{S^S} \left( x_0; u^A, u^S, u^T \right) \right\}
\]

**Coalitive Pareto decision parameterization**

For given \( u^A \in U^A \) and \( u^T \in U^T \), \( \hat{u}_C^S \in U^S_C \) is efficient \( \iff \) There exists \( \bar{\xi}^S \in \mathbb{R}^{S^S} \) and \( \bar{\xi}^S > 0 \) such that \( \hat{u}_C^S \) is a Pareto optimal solution of the single-objective problem

\[
\arg\min_{u^A \in U^A, u^S_C \in U^S_C, u^T \in U^T} \left\{ J \left( x_0; u^A, u^S_C, u^T \right) \triangleq \sum_{i=1}^{S^S} \xi_i^S J_i \left( x_0; u^A, u^S_C, u^T \right) \right\}
\]

Know-How-to-Cooperate: Managing Interference & Facilitating Between Goals
Balance of Intra-Team and Inter-Team Objectives

Let $Q_f \triangleq \sum_{i=1}^{S_f} \xi_i^S Q_{if}; \quad Q \triangleq \sum_{i=1}^{S} \xi_i^S Q_i; \quad R^{SA}_j \triangleq \sum_{i=1}^{S} \xi_i^S R^{SA}_j; \quad R^{SS}_j \triangleq \sum_{i=1}^{S} \xi_i^S R^{SS}_j; \quad R^{ST}_j \triangleq \sum_{i=1}^{S} \xi_i^S R^{ST}_j$

$J(x_0; u^A(\cdot), u^S_C(\cdot), u^T) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} [x^T(\tau)Qx(\tau)] d\tau$

$+ \int_{t_0}^{t_f} \left[ \sum_{j=1}^{S} (u^S_{Cj}(\tau))^T R^{SS}_j u^S_C(\tau) - \sum_{j=1}^{m_A} (u^A_j(\tau))^T R^{SA}_j u^A_j(\tau) - \sum_{j=1}^{m_T} (u^T_j(\tau))^T R^{ST}_j u^T_j(\tau) \right] d\tau$

Further $R^{SS} \triangleq \text{diag}(R^{SS}_1, \ldots, R^{SS}_S); \quad R^{SA} \triangleq \text{diag}(R^{SA}_1, \ldots, R^{SA}_{m_A}); \quad R^{ST} \triangleq \text{diag}(R^{ST}_1, \ldots, R^{ST}_{m_T})$

Know-How-to-Cooperate: Inducing Pareto Coordination
A Decision Architecture for Risk Averse based Multi-Agent Differential Game of A Kind

Performance Information System

Risk Aversion

Risk Modeling
Risk Measures

Joint Utility Modeling
Probing and Caution

Risk Averse Decision Policies

Autonomous Interaction Process
Performance Information:
Perception of Elements in Current Situation

**Status**
- Interactive teams A, S, and T of members
- End game engagement near assets being either tracked or protected

**Attributes**
- Decision horizon \( t \in [t_0, t_f] \)
- Default information of dynamical features of interaction management
  \[ A, B^A, B^S, B^T \text{ and } Q_f \geq 0; Q \geq 0; R^{SS}, R^{SA}, R^{ST} > 0 \]
- Pairs \((A, B^A), (A, B^S)\) and \((A, B^T)\) stabilizable

**Environmental Features**
- Stationary Wiener process \((\Omega, F, \{F_t\}_{t \geq t_0 \geq 0}, P)\) for the uncertain environment which affects the outcomes via its mixed random sample path realizations

\[
E \left\{ w(\tau) - w(\sigma) \right\} \left[ w(\tau) - w(\sigma) \right]^T \right\} = W |\tau - \sigma|; \quad \tau, \sigma \in [t_0, t_f].
\]
Performance Information: Comprehension of the Current Situation

Multi-Agent Differential Game
- forms action-outcome pictures

\[
dx(t) = (Ax(t) + B^A u^A(t) + B^S u^S_C(t) + B^T u^T(t))dt + Gdw(t) \\
x(t_0) = x_0
\]

\[
J(x_0; u^A(\cdot), u^S_C(\cdot), u^T) = x^T(t_f)Q_f x(t_f) + \int_{t_0}^{t_f} [x^T(\tau)Q x(\tau)] d\tau \\
+ \int_{t_0}^{t_f} \left[ (u^S_C)^T(\tau)R^{SS} u^S_C(\tau) - (u^A)^T(\tau)R^{SA} u^A(\tau) - (u^T)^T(\tau)R^{ST} u^T(\tau) \right] d\tau
\]

- comprehend the significance of linear-quadratic nature of interaction dynamics and integrate this characteristic property to the performance-measure

\[
\Rightarrow J(x_0; u^A(\cdot), u^S_C(\cdot), u^T) \text{ is a random variable with Chi-squared type !!}
\]
Performance Information: Information System and Its Value

Multi-Agent Differential Game
• determines which cues are relevant to performance uncertainty and risk, e.g.,

Moment- and Cumulant-Generating Functions (parameterized by $\theta$)

\[
\varphi(s, x_s; \theta) \triangleq E \left\{ \exp \left\{ \theta J(s, x_s) \right\} \right\}
\]
\[
\psi(s, x_s; \theta) \triangleq \ln \left\{ \varphi(s, x_s; \theta) \right\}
\]

$\varphi(s, x_s; \theta)$ denotes information system, $\varphi$

$\psi(s, x_s; \theta)$ denotes the value of information system, $\psi$

$\theta \in \Theta$ denotes the set of parameters for the information system, $\varphi$

$\Omega$ denotes the set of uncertain states of the uncertain environment

Transferring Data to States of Knowledge
$U^A$, $U^S_C$, and $U^T$ denote the sets of actions via state feedback strategies to maintain a fair degree of accuracy on team behaviors and the security level utility.

\[
\begin{align*}
\gamma^A : \Gamma^A &\mapsto U^A \\
\gamma^S : \Gamma^S &\mapsto U^S_C \\
\gamma^T : \Gamma^T &\mapsto U^T \\
 u^A &= \gamma^A(\eta) \\
 u^S_C &= \gamma^S(\eta) \\
 u^T &= \gamma^T(\eta) \\
 \eta &= (t, x(t))
\end{align*}
\]

Closed-Loop Decisions

Decision Variables Controlled by Autonomous Teams
Performance Information:
Information Value Determinant → Outcome Function

\( f \) denotes the outcome function mapping outcome-action-uncertain state triplets into outcomes, e.g., \( x \in L^2 \{ F_t \}_{s \leq t \leq f} \left( \Omega, C \left( [s, t_f] ; R^n \right) \right) \) of \( R^n \) – valued, square integrable processes on \([s, t_f]\) that are adapted to the sigma field \( F_t \) generated by \( w(t) \), e.g.,

\[
dx(t) = f\left(x, u^A, u^S, u^T ; w \right); \quad x(s) = \left(A + B^A K^A + B^S K^S + B^T K^T \right) x(t) dt + Gd\omega(t); \quad x(s)
\]

In a compact notation, it results in

\[
dx(t) = F(t)x(t) dt + Gd\omega(t); \quad x(s)
\]

\[
F(t) \triangleq A + B^A K^A(t) + B^S K^S(t) + B^T K^T(t)
\]
Performance Information: Information Value Determinant $\rightarrow$ Utility Function

$J(s,x_s)$ denotes an utility function mapping outcomes into utility levels i.e.,

$$J(s,x_s) \equiv x^T(t_f)Q_f x(t_f) + \int_s^{t_f} x^T(\tau) \left[ Q + \left( K^S \right)^T(\tau) R^{SS} K^S(\tau) \right] x(\tau) d\tau$$

$$+ \int_s^{t_f} x^T(\tau) \left[ -\left( K^A \right)^T(\tau) R^{SA} K^A(\tau) - \left( K^T \right)^T(\tau) R^{ST} K^T(\tau) \right] x(\tau) d\tau$$

Or, equivalently

$$J(s,x_s) = x^T(t_f)Q_f x(t_f) + \int_s^{t_f} x^T(\tau) N(\tau) x(\tau) d\tau$$

$$N(t) \equiv Q + \left( K^S \right)^T(t) R^{SS} K^S(t) - \left( K^A \right)^T(t) R^{SA} K^A(t) - \left( K^T \right)^T(t) R^{ST} K^T(t)$$

Stochastic Preference for Autonomous Teams
Performance Information: Attributes of Information Value \( \rightarrow \) Cumulants of Performance-Measure

\[
\psi (s, x_s; \theta) = \sum_{k=1}^{\infty} \frac{\partial^{(k)}}{\partial (\theta)^{(k)}} \psi (s, x_s; \theta) \bigg|_{\theta=0} \frac{(\theta)^k}{k!} \triangleq \sum_{k=1}^{\infty} \frac{(\theta)^k}{k!} \kappa_k
\]

\[
\kappa_k = \left(x_s\right)^T \frac{\partial^{(k)}}{\partial (\theta)^{(k)}} Y(s; \theta) \bigg|_{\theta=0} x_s + \frac{\partial^{(k)}}{\partial (\theta)^{(k)}} v(s; \theta) \bigg|_{\theta=0} \triangleq \left(x_s\right)^T H(s, k) x_s + D(s, k)
\]

\[
\frac{d}{ds} Y(s; \theta) = -\left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right]^T Y(s; \theta)
\]

\[
-\gamma(s; \theta) \left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right] - 2 Y(s; \theta) G W G^T Y(s; \theta)
\]

\[
-\theta \left[ Q + \left(K^S\right)^T (s) R^{SS} K^S (s) - \left(K^A\right)^T (s) R^{SA} K^A (s) - \left(K^T\right)^T (s) R^{ST} K^T (s) \right]
\]

\[
\frac{d}{ds} v(s; \theta) = -Tr \left\{ Y(s; \theta) G W G^T \right\}
\]

\[
Y(t_f; \theta) = \theta Q_f; \quad v(t_f; \theta) = 0.
\]
Performance Information: 
Information Dynamics of Performance-Measure

\[
\frac{d}{ds} H(s,1) = - \left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right]^T H(s,1) \\
- H(s,1) \left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right] \\
- Q - \left( K^S \right)^T(s) R^{SS} K^S (s) + \left( K^A \right)^T(s) R^{SA} K^A (s) + \left( K^T \right)^T(s) R^{ST} K^T (s)
\]

\[
\frac{d}{ds} H(s,r) = - \left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right]^T H(s,r) \\
- H(s,r) \left[ A + B^A K^A (s) + B^S K^S (s) + B^T K^T (s) \right] \\
- \sum_{v=1}^{r-1} \frac{2r!}{v!(r-v)!} H(s,v) GWG^T H(s,r-v), \quad 2 \leq r \leq k
\]

\[
\frac{d}{ds} D(s,r) = -Tr \left\{ H(s,r) GWG^T \right\}, \quad D(t_f,r) = 0, \quad 1 \leq r \leq k
\]

\[
H(t_f,1) = Q_f; \quad H(t_f,r) = 0 \text{ for } 2 \leq r \leq k
\]
A Decision Architecture for Risk Averse based Multi-Agent Differential Game of A Kind

SENSE
Performance Information System

DECIDE
Risk Aversion
Role
Action
Risk Modeling
Risk Measures
Behavior
Joint Utility Modeling
Probing and Caution
Execution
Risk Averse Decision Strategy Implementation

ACT

Autonomous Interaction Process

Performance Information System
Risk Averse Decision Policies
Decision Making under Uncertainty: A Risk-Value Model

Suppose $S_i$ and $S_j$ are two stochastic systems. A preference comparison between $S_i$ and $S_j$ can be made by a risk-value model:

$$S_i > S_j \quad \text{if and only if} \quad \phi(V(S_i), R(S_i)) \geq \phi(V(S_j), R(S_j))$$

where

- $V$ measures the value of a system,
- $R$ measures its riskiness, and
- $\phi$ reflects the trade-off between value and riskiness.

Models of Perceived Risk:

- **Finance** $\rightarrow$ Variance
- **Safety** $\rightarrow$ Both the *probability* and the *magnitude* of adverse effects

*Models of Risk Must be Specialized by Classes of Applications*
Decision Making under Uncertainty: Measures of Performance Risk

Some Possible Interpretations

• **LQG Control** → An expected utility minimizer with risk defined as expected values
• **Risk Sensitive Control** → A risk sensitivity minimizer with all centralized moments weighted in a specific way to yield the riskiness
Decision Making under Uncertainty: A General Measure of Performance Risk

- Physical/social fuel; energy; dollars
- Intermediate path "in-between"; "middle-man"
- Idea: Pursue additional goals...

- LQG Control
- Risk Sensitivity & Beyond...

In Statistical Control: Weightings for Mean, Variance, Skewness, Flatness,... → Design Freedom
Decision Making under Uncertainty: Preference Model for Decision Making

**Fundamental Issue:** Drive some state $x \left( x_0, w, u^A_C, u^S_C, u^T_C \right)$ to zero

**New Model of Preference:** *Statistical Control*

$$
\min_{K^A, K^S, K^T} \left\{ \sum_{r=1}^{k} \mu_r K_r \left( K^A, K^S, K^T \right) \right\}
$$

Emphases are on different competing cumulants and team prioritizations which are determined by $\mu \triangleq \left\{ \mu_r \geq 0 \right\}_{r=1}^{k}$ with $\mu_r > 0$.

**Classical Preference:** *LQG Control*

$$
\min_{K^A, K^S, K^T} \left\{ \kappa_1 \left( K^A, K^S, K^T \right) \right\}
$$

**Fairly New Preference:** *Risk-Sensitive Control*

$$
\min_{K^A, K^S, K^T} \left\{ \sum_{r=1}^{\infty} \frac{\theta}{r!} \kappa_r \left( K^A, K^S, K^T \right) \right\}
$$
Decision Making under Uncertainty: Performance Uncertainty Probing & Cautioning

\[ \phi_0 : \{t_0\} \times \left( \mathbb{R}^{n \times n} \right)^k \times \mathbb{R}^k \mapsto \mathbb{R}_+ \]

\[ F \triangleq F_1 \times \cdots \times F_k \]
\[ G \triangleq G_1 \times \cdots \times G_k \]

\[ H_r = H_r \left( ; K^A, K^S, K^T \right) \]
\[ D_r = D_r \left( ; K^A, K^S, K^T \right) \]

\[ H_f = Q_f \times 0 \times \cdots \times 0 \]
\[ D_f = 0 \times \cdots \times 0 \]

\[ \frac{d}{ds} H(s) = F(s; H(s); K^A(s), K^S(s), K^T(s)); \quad H(t_f) = H_f \]

\[ \frac{d}{ds} D(s) = G(s; H(s)); \quad D(t_f) = D_f \]

\[ \phi_0 \left( t_0; H, D \right) \triangleq \mu_1 K_1 + \mu_2 K_2 + \mu_3 K_3 + \mu_4 K_4 + \cdots + \mu_k K_k \]
\[ = \mu_1 \left[ x_0^T H_1(t_0) x_0 + D_1(t_0) \right] + \cdots + \mu_k \left[ x_0^T H_k(t_0) x_0 + D_k(t_0) \right] \]
Decision Making under Uncertainty: Principle of Optimality

An optimal strategy has the property that whatever the initial state and time, all remaining decisions must constitute an optimal strategy…

Classical Control Class $\Rightarrow$ Terminal Cost Problems

Statistical Control Class $\Rightarrow$ The Initial Cost Problem
Decision Making under Uncertainty: Tenet of Transition

“…we are dealing with a family of optimization based on different starting points. Consider an interlude of time in mid-play. At its commencement the path has reached some definitive point. Consider all possible \( x(H,D) \) which may be reached at the end of the interlude for all possible choices of \( \left( u^A, u^S_C, u^T \right)(K^A, K^S, K^T) \).

Suppose that for each endpoint, the optimization beginning there has already been solved (\( V \) is known there). Then the payoff resulting from each choice of \( \left( u^A, u^S_C, u^T \right)(K^A, K^S, K^T) \) will be known, and they are to be so chosen as to render it minimum. When we let the duration of the interlude approach \( t_0 \left( t_f \right), …\)”

**Terminal Cost Problems**

\[ x(t) \]

\[ V(t + \Delta t, x(t + \Delta t)) \]

\[ t_0 \quad t \quad t + \Delta t \quad t_f \]

**The Initial Cost Problem**

\[ V(s - \Delta s, H(s - \Delta s), D(s - \Delta s)) \]

\[ t_0 \quad s - \Delta s \quad s \quad t_f \]
“... we are dealing with a family of optimization based on different starting points. Consider an interlude of time in mid-play. At its commencement the path has reached some definitive point. Consider all possible \( X (H, D) \) which may be reached at the end of the interlude for all possible choices of \( (u^A, u^C, u^T) ( K^A, K^S, K^T ) \).

Suppose that for each endpoint, the optimization beginning there has already been solved \((V \text{ is known there})\). Then the payoff resulting from each choice of \( (u^A, u^C, u^T) ( K^A, K^S, K^T ) \) will be known, and they are to be so chosen as to render it minimum. When we let the duration of the interlude approach \( t_0 ( t_f ) \),...”

\[ \Rightarrow \text{Leads to as sufficient condition to HJB equation} \]

\[
\begin{align*}
(Y, Z) & \quad V(\varepsilon, Y, Z) & \quad \varepsilon & \quad W(\varepsilon, Y, Z) \\
& \quad t_0 & \quad \varepsilon & \quad t_f \\
\text{Value function} & \quad (\text{Cost-to-Go}) & \quad \text{Cost-to-Come}
\end{align*}
\]
Decision Making under Uncertainty: 
HJB Equation for Mayer Problem

\[
\begin{align*}
\min_{K^A \in \mathbb{R}^A, K^S \in \mathbb{R}^S, K^T \in \mathbb{R}^T} & \quad \frac{\partial}{\partial \epsilon} V(\epsilon, Y, Z) + \frac{\partial}{\partial \text{vec}(Y)} V(\epsilon, Y, Z) \cdot \text{vec} \left( F(\epsilon, Y, K^A, K^S, K^T) \right) \\
& + \frac{\partial}{\partial \text{vec}(Z)} V(\epsilon, Y, Z) \cdot \text{vec} \left( G(\epsilon, Y) \right)
\end{align*}
\]

B.C.
Decision Making under Uncertainty: Solving the Mayer Problem

Candidate Solution for HJB equation

\[
W(\varepsilon, Y, Z) = x_0^T \sum_{r=1}^{k} \mu_r (Y_r + E_r(\varepsilon)) x_0 + \sum_{r=1}^{k} \mu_r (Z_r + T_r(\varepsilon))
\]

\[E \in C^1([t_0, t_f]; R^{n \times n}) \text{ and } T \in C^1([t_0, t_f]; R) \text{ yet to be determined}\]

Associated HJB equation

\[
\min_{\kappa^A \in \mathbb{R}^A, \kappa^S \in \mathbb{R}^S, \kappa^T \in \mathbb{R}^T} \left\{ x_0^T \left[ \sum_{r=1}^{k} \mu_r \frac{d}{d\varepsilon} E_r(\varepsilon) \right] x_0 + \sum_{r=1}^{k} \mu_r \frac{d}{d\varepsilon} T_r(\varepsilon) + x_0^T \left[ \sum_{r=1}^{k} \mu_r F_r(\varepsilon, Y, K^A, K^S, K^T) \right] x_0 + \sum_{r=1}^{k} \mu_r G_r(\varepsilon, Y) \right\} = 0
\]
Decision Making under Uncertainty: Risk-Averse and Pareto Optimal Decision Strategies

\[ K^{S^*}(s) = -\left( R^{SS} \right)^{-1} (B^s)^T \sum_{r=1}^{k} \hat{\mu}_r H_r^*(s) \]

\[ K^{A^*}(s) = \left( R^{SA} \right)^{-1} (B^A)^T \sum_{r=1}^{k} \hat{\mu}_r H_r^*(s) \]

\[ K^{T^*}(s) = \left( R^{ST} \right)^{-1} (B^T)^T \sum_{r=1}^{k} \hat{\mu}_r H_r^*(s) \]

\[ \hat{\mu}_r \triangleq \frac{\mu_r}{\mu_1} \]

\[ x^* \triangleq \begin{bmatrix} (x^{A^*})^T & (x^{S^*})^T & (x^{T^*})^T \end{bmatrix}^T \]

\[ K^{A^*} \triangleq \begin{bmatrix} K^{AA}_* & K^{AS}_* & K^{AT}_* \end{bmatrix}; \quad K^{S^*} \triangleq \begin{bmatrix} K^{SA}_* & K^{SS}_* & K^{ST}_* \end{bmatrix}; \quad K^{T^*} \triangleq \begin{bmatrix} K^{TA}_* & K^{TS}_* & K^{TT}_* \end{bmatrix} \]

\[ u^{A^*}(t) = K^{AA}_*(t)x^{A^*}(t) + K^{AS}_*(t)x^{S^*}(t) + K^{AT}_*(t)x^{T^*}(t) \]

\[ u^{S^*}(t) = K^{SA}_*(t)x^{A^*}(t) + K^{SS}_*(t)x^{S^*}(t) + K^{ST}_*(t)x^{T^*}(t) \]

\[ u^{T^*}(t) = K^{TA}_*(t)x^{A^*}(t) + K^{TS}_*(t)x^{S^*}(t) + K^{TT}_*(t)x^{T^*}(t) \]

Adaptable and Robust Against Adversaries and Random Sample-Path Realizations
"No Free Lunch" Theorem: Robustness vs. Efficiency
Generality vs. Depth

Performance Uncertainty Observer

Risk-Averse and Pareto Optimal Strategies

\[ \left( R^{ST} \right)^{-1} \left( B^T \right)^T \]

\[ - \left( R^{SS} \right)^{-1} \left( B^S \right)^T \]

\[ \left( R^{SA} \right)^{-1} \left( B^A \right)^T \]
Multi-Level Control and Coordination

Risk-Averse Yet Pareto Efficient Coordinators

Aggressor Team

Sensor Team

Target Team

\[ \alpha^s, \xi^s, \mu \]

\[ u^A \]

\( (x^A, x^S, x^T) \)

\[ u^S \]

\( (x^A, x^S, x^T) \)

\[ u^T \]

\( (x^A, x^S, x^T) \)

Confrontation Dynamics
Unfolding Interactions
Conclusions

Major contributions offered by statistical game theory to risk-averse and linear-quadratic multi-agent differential games, include:

- Performance information modeling
- Determinants of performance information value
- Cumulants having consistent effects on performance information value
- Attributes of decision settings and decision makers not having the same effects on performance information value
- Natural linkages to LQG theory and risk sensitive control
- In-depth knowledge of utilizing “performance-measure statistics” to shape team performance robustness
- Integration of perceived performance risk with team decision strategies, and
- Unifying framework for potential streams of research on performance risk judgments and decision making

Performance uncertainty representations and adaptive algorithms from logic and probability are combined to maintain performance uncertainty representations that are compact and robust.