Multiplicative Stochastic Control Systems

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1. Introduction

\[ P_n := (0, \infty)^n \]

\[ \mathcal{M}_n := \{ f: P_n \to P_n \mid f \text{ is monotone & homogeneous} \} \]

\[ x \leq y \Rightarrow f(x) \leq f(y), \quad f(\lambda x) = \lambda f(x), \quad \lambda > 0, x \in P_n \]

**Eigenvalue problem:** To find verifiable conditions ensuring the existence of a pair \((\lambda, x) \in (0, \infty) \times P\) such that \(\lambda\) is an eigenvalue of \(f\) and \(x\) is an eigenvector corresponding to \(\lambda\)

\[ f(x) = \lambda x \]

**The Perron-Frobenius theorem:** If \(f\) is a linear function associated with a nonnegative and communicating matrix, then \(f\) has a unique positive eigenvalue \(\lambda(f)\) and, moreover, two different eigenvectors associated with \(\lambda(f)\) are linearly dependent (Seneta, 1980, Minc, 1988).

**The main objective:** To formulate generalized communication conditions, based on the idea of communicating matrix and applicable to a general monotone and homogeneous function \(f\), under which the existence of a (necessarily unique) non-linear eigenvalue \(\lambda(f)\) can be ensured.
Motivation: • $X_0, X_1, X_2, \ldots$ a Markov chain with state space $S = \{1, 2, \ldots, n\}$

$$P[X_{t+1} = y \mid X_0, X_1, \ldots, X_t = x] = p_{xy}$$

$C: S \to \mathbb{R}$ the one-step cost function

For a risk-averse observer, the (long-run) average cost is

$$J(x) = \lim_{t \to \infty} \frac{1}{t} \log \left( E_x \left[ e^{\sum_{k=0}^{t-1} C(X_k)} \right] \right)$$

The Poisson equation

$$e^g e^{h(i)} = e^{C(i)} \sum_j p_{ij} e^{h(j)} \quad \Rightarrow \quad g = J(\cdot)$$

$$f(z) = (e^{C(i)} \sum_j p_{ij} z_j ; i = 1, 2, \ldots, n)', \quad x = (e^{h(1)}, \ldots, e^{h(n)})'$$

$$e^g x = f(x)$$
• Controlled Case: $U$ set of possible actions, $A_t$ action applied at time $t$

$$P[X_{t+1} = y | X_0, A_0, X_1, A_1 \ldots, X_{t-1}, A_{t-1} X_t = x, A_t = a] = p_{xy}(a)$$

$C(x, a)$ is the cost incurred at state $x$ when action $a$ is applied

The strategy $\pi$ for choosing actions may depend on the states observed up to the present and on the actions previously applied:

$$J(x, \pi) = \lim_{t \to \infty} \frac{1}{t} \log \left( E_x^\pi \left[ e^{\sum_{k=0}^{t-1} C(X_k, A_k)} \right] \right)$$

$$J(x) = \min_{\pi} J(x, \pi)$$

The optimality equation

$$e^g e^{h(i)} = \min_a \left[ e^{C(i,a)} \sum_j p_{ij}(a) e^{h(j)} \right] \quad \Rightarrow \quad g = J(\cdot)$$

$$f(z) = \left( \min_a \left[ e^{C(i,a)} \sum_j p_{ij}(a) z_j \right] ; \quad i = 1, 2, \ldots, n \right) ', \quad x = (e^{h(1)}, \ldots, e^{h(n)})'$$

$$e^g x = f(x), \quad f \in \mathcal{M\mathcal{H}_n}$$
2. The Proper Value

Collatz-Wielandt relations (Minc, 1988, Gaubert and Gunawardena, 2004):

\[
\lambda^+(f) = \inf_{z \in P_n} \max_i \frac{f_i(z)}{z_i} i = 1, 2, \ldots, n;
\]

\[
\lambda^-(f) = \sup_{z \in P_n} \min_i \frac{f_i(z)}{z_i} i = 1, 2, \ldots, n. \tag{2.1}
\]

- A proper value is unique:

\[
f(x) = \lambda x, \quad x \in P_n \Rightarrow \lambda = \lambda(f) = \lambda^+(f) = \lambda^-(f).
\]

\[
\mathcal{E}(f) := \{x | f(x) = \lambda(f)x\}
\]

- There may be linearly independent eigenvectors

\[
f(x) = (x_1 \land 2x_2, x_2 \land 2x_1)'
\]

\[
\lambda(f) = 1, \quad \mathcal{E}(f) = \{(x_1, x_2)' | x_2 \leq 2x_1 \leq 4x_2\}
\]
Hilbert's distance: \( d_H(x, y) = \log \left( \frac{\max \{ x_k/y_k \}}{\min \{ x_r/y_r \}} \right) \), \( x, y \in \mathcal{P}_n \)

Definition 2.1. (i) A non-empty set \( B \subset \mathcal{P}_n \) is projectively bounded if

\[
\sup_{y \in B} d_H(y, 1) < \infty, \quad 1 = (1, 1, \ldots, 1)'
\]

\[
\sup_{y \in B} \frac{\max(y)}{\min(y)} < \infty, \quad \min(y) = \min \{ y_1, \ldots, y_n \}, \quad \max(y) = \max \{ y_1, \ldots, y_n \}
\]

(iii) \( B \subset \mathcal{P}_n \) is \( f \)-invariant if \( x \in B \Rightarrow f(x) \in B \).

Theorem 2.1. For each \( f \in \mathcal{M} \mathcal{H}_n \) the following conditions (i) and (ii) are equivalent:

(i) The function \( f \) has an eigenvalue, that is, there exists a pair \( (\lambda, y) \in (0, \infty) \times \mathcal{P}_n \) such that \( f(y) = \lambda y \).

(ii) There exists a nonempty set \( B \subset \mathcal{P}_n \) which is projectively bounded and invariant with respect to \( f \).
Definition 2.2. The upper and lower eigenspaces associated with the pair \((f, a)\):

\[ S^a(f) := \{ x \in \mathcal{P}_n \mid f(x) \leq ax \} \quad \text{and} \quad S_a(f) := \{ x \in \mathcal{P}_n \mid f(x) \geq ax \}, \]

respectively. The upper and lower eigenspaces \(S^a(f)\) and \(S_a(f)\) are \(f\)-invariant.

The Linear Case. Let \(A = [A_{ij}]\) be a nonnegative matrix of order \(n \times n\) with non-null rows, and define

\[ f_A(x) := Ax, \quad x \in \mathcal{P}_n, \quad (2.2) \]

Definition 2.3. \(A\) is communicating if the following condition holds: For each \(i, j \in \{1, 2, \ldots, n\}\) there exist \(i_1, \ldots, i_k \in \{1, 2, \ldots, n\}\) satisfying

\[ i_0 = i, \quad i_k = j \quad \text{and} \quad A_{i_{r-1}i_r} > 0, \quad r = 1, 2, \ldots, k. \quad (2.3) \]

Theorem 2.2. [Perron-Frobenius] If \(A\) is communicating

(i) The function \(f_A\) has an eigenvalue: \(f_A(y) = \lambda y, \quad (\lambda, y) \in (0, \infty) \times \mathcal{P}_n.\)

(ii) If \((\lambda_1, y_1) \in (0, \infty) \times \mathcal{P}_n\) satisfies \(f_A(y_1) = \lambda_1 y_1, \) then \(\lambda = \lambda_1\) and \(y = cy_1\) for some \(c > 0.\)

If $A$ is a nonnegative matrix, then

(a) The property of being communicating does not depend on the exact values of the entries of $A$, but just on which components of $A$ are non-null, and

(b) The positivity of an entry of matrix $A$ is characterized by

$$A_{i,j} > 0 \iff \lim_{t \to \infty} f_{A,i}(te_j + 1) = \lim_{t \to \infty} \left[ A_{i,j}t + \sum_{k=1}^{n} A_{i,k} \right] = \infty.$$

**Definition 2.4.** For a given function $f \in M\mathcal{H}_n$, the corresponding (communication) matrix $M(f) \equiv [M(f)_{i,j}]$ of order $n \times n$ is defined as follows: For $i, j = 1, 2, \ldots, n$,

$$M(f)_{i,j} := \begin{cases} 1, & \text{if } \lim_{t \to \infty} f_i(te_j + 1) = \infty, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$
Theorem 2.3. [Gaubert and Gunawerdena, 2004]. Let $f \in \mathcal{MH}_n$ be such that the matrix $M(f)$ is communicating.

(i) If $S^a(f) \neq \emptyset$ then $S^a(f)$ is projectively bounded.

(ii) The function $f$ has an eigenvalue: $f(y) = \lambda y$, $(\lambda, y) \in (0, \infty) \times \mathcal{P}_n$, where $\lambda \equiv \lambda(f)$ is uniquely determined by (2.1). Moreover, the corresponding eigenspace $\mathcal{E}(f)$ is projectively bounded.

- $r \in \mathcal{P}_n$, $f \in \mathcal{MH}_n$

\[
rf(x) = (r_1f_1(x), \ldots, r_nf_n(x))'
\]

\[
rf(x) = \lambda_rx, \quad (\lambda_r, x) \in (0, \infty) \times \mathcal{P}_n
\]

$C_w$ is the class of all functions in $\mathcal{MH}_n$ satisfying the following weak convexity condition: If $f_i(x)$ depends on $x_j$, then there exist $a_{ij} > 0$ and $b_{ij} \in \mathbb{R}$ such that

\[
f_i(te_j + 1) \geq a_{ij}t + b_{ij}
\]

$f_A \in C_w$, $f \in C_w$ & $f_\alpha(x) = f(x^\alpha)^{1/\alpha} \Rightarrow f_\alpha \in C_w$, $\alpha > 0$
Theorem 2.4. [Cavazos-Cadena and Hernández-Hernández, 2008] If \( f \in \mathcal{M} \mathcal{H}_n \) is weakly convex, then the following conditions (i) and (ii) are equivalent:

(i) For each \( r \in \mathcal{P}_n \) the Poisson equation

\[
r f(x) = \lambda x
\]

has a solution \( (\lambda, x) \in (0, \infty) \times \mathcal{P}_n \).

(ii) The matrix \( M(f) \) has a single communicating class \( \mathcal{C}(f) \) and \( S \setminus \mathcal{C}(f) \) can be partitioned as

\[
S \setminus \mathcal{C}(f) = T_1 \cup T_2 \cup \cdots \cup T_r
\]

where

\[
i \in T_k \Rightarrow f_i(\cdot) \text{ is a function of } x_s, \ s \in T_{k-1} \cup \cdots \cup T_0, \ T_0 = \mathcal{C}(f).
\]
3. A Generalized Communication Condition

\[ f_A(x) = Ax \]

Lemma 3.1. Let \( A \) be a nonnegative matrix of order \( n \times n \) whose rows are non-null, and let \( f_A(\cdot) \) be as in (2.2). In this context, properties (i)-(iii) below are equivalent.

(i) \( A \) is communicating.

(ii) \( B := I + A + A^2 + \cdots + A^{n-1} > 0 \).

(iii) For each \( i, j \in \{1, 2, \ldots, n\} \),

\[ \lim_{t \to \infty} f_{B,i}(te_j + 1) = \lim_{t \to \infty} \sum_{k=0}^{n-1} f_{A,i}^k(te_j + 1) = \infty. \]

Assumption 3.1. For each \( i, j \in 1, 2, \ldots, n \),

\[ \lim_{t \to \infty} \sum_{k=0}^{n-1} f_i^k(te_j + 1) = \infty. \] (3.1)

\[ \lim_{t \to \infty} \bigvee_{k=0}^{n-1} f_i^k(te_j + 1) = \infty, \quad \lim_{t \to \infty} \bigvee_{k=0}^{n-1} f_i^k(te_j + 1_{\{j\}^c}) = \infty. \]

\[ x_{D,i} = x_i, \quad i \in D, \quad x_{D,i} = 0, \quad i \in D^c \]
Theorem 3.1. For each $f \in \mathcal{MH}_n$ the following assertion is valid:

If the matrix $M(f)$ is communicating then Assumption 3.1 is satisfied by $f$;

see Definitions 2.3 and 2.4.

Example 3.1.

\[
\begin{align*}
  f(x) = & \begin{bmatrix}
    a_1 x_1 \lor a_2 x_2 \\
    (b_1 x_1 \lor b_2 x_3) + (b_3 x_2 \lor b_4 x_3) \\
    c_1 x_1 \land c_2 x_2
  \end{bmatrix}, \\
  M(f) = & \begin{bmatrix}
    1 & 1 & 0 \\
    1 & 1 & 1 \\
    0 & 0 & 0
  \end{bmatrix}
\end{align*}
\]  

$M(f)$ is not communicating, since its third row is null. $f$ satisfies Assumption 3.1.  \(\square\)
4. Projective Boundedness of Upper Eigenspaces

Theorem 4.1. Let $f \in \mathcal{MH}_n$ and $a > 0$ be arbitrary and suppose that Assumption 3.1 holds. In this case

(i) If $S^a(f) \neq \emptyset$ then this set is is projectively bounded.

Consequently,

(ii) The function $f$ has a unique eigenvalue $\lambda \equiv \lambda(f)$, and the corresponding eigenspace $\mathcal{E}(f)$ is projectively bounded.

Proof. \[ F := \sum_{k=0}^{n-1} f^k \in \mathcal{MH}_n \]

\[ S^b(F) \neq \emptyset \Rightarrow S^b(F) \text{ is projectively bounded.} \quad (4.1) \]

\[ x \in S^a(f) \Rightarrow f(x) \leq ax \Rightarrow f^k(x) \leq a^k x, \quad k = 0, 1, 2, \ldots, \]

\[ \Rightarrow x \in S^b(F), \quad \text{where } b := \sum_{k=1}^{n-1} a^k, \]

and then

\[ S^a(f) \subset S^b(F). \]
\[
\lim_{t \to \infty} F_i(t e_j + 1) = \infty, \quad i, j = 1, 2, \ldots, n. \quad (4.2)
\]

Assume that \( S^b(F) \) is not projectively bounded:

\[
\{ x^k \} \subset S^b(F) \quad \text{and} \quad \frac{\max(x^k)}{\min(x^k)} \to \infty \quad \text{as} \quad k \to \infty.
\]

\( y_k := (\min(x^k))^{-1} x^k \) it follows that

\[
\{ y^k \} \subset S^b(F), \quad \min(y^k) = 1 \quad \text{and} \quad \max(y^k) = \frac{\max(x^k)}{\min(x^k)} \to \infty \quad \text{as} \quad k \to \infty.
\]

\[
\min(y^k) = y_{i(k)}^k, \quad \max(y^k) = y_{j(k)}^k, \quad j(k) = j^*, \quad i(k) = i^*.
\]

\[
1 = \min(y^k) = y_{i^*}^k, \quad \max(y^k) = y_{j^*}^k, \quad k = 1, 2, 3, \ldots, \quad \lim_{k \to \infty} y_{j^*}^k = \infty. \quad (4.3)
\]

\[
y^k \geq [y_{j^*}^k - 1] e_{j^*} + 1 \Rightarrow F([y_{j^*}^k - 1] e_{j^*} + 1) \leq F(y^k) \leq b y^k
\]

\[
F_{i^*}([y_{j^*}^k - 1] e_{j^*} + 1) \leq b y_{i^*}^k = b
\]

\[
\lim_{t \to \infty} F_{i^*}(t e_{j^*} + 1) = \lim_{k \to \infty} F_{i^*}([y_{j^*}^k - 1] e_{j^*} + 1) \leq b.
\]

**Example 3.1** [Continued.] The function \( f \) satisfies Assumption 3.1, so that \( f \) has an eigenvalue, by Theorem 4.1 \( \square \)
5. The Dual Function

**Definition 5.1.** Given $f \in \mathcal{M} \mathcal{H}_n$, the corresponding dual function $\tilde{f} : \mathcal{P}_n \to \mathcal{P}_n$ is defined as follows:

$$\tilde{f}(y) := f(y^{-1})^{-1}, \quad y \in \mathcal{P}_n$$

$$y^{-1} := (y_1^{-1}, \ldots, y_n^{-1})'. $$

\[\tilde{f} \in \mathcal{M} \mathcal{H}_n, \quad \tilde{f} = f, \quad \tilde{f}(g) = \tilde{f}(\tilde{g}), \quad \tilde{f} \lor g = \tilde{f} \land \tilde{g} \quad \text{and} \quad \tilde{f} \land g = \tilde{f} \lor \tilde{g}. \quad (5.1)\]

$$B^{-1} := \{ x^{-1} | x \in B \}, \quad B \subset \mathcal{P}_n$$

$$\sup_{y \in B^{-1}} \frac{\max(y)}{\min(y)} = \sup_{x \in B} \frac{\max(x)}{\min(x)} ,$$

so that

$$B \text{ is projectively bounded } \iff B^{-1} \text{ is projectively bounded;} \quad (5.2)$$
\[ S_a(f) = \left[ S^{a-1}(\tilde{f}) \right]^{-1}. \]

**Theorem 5.1.** Let \( f \in \mathcal{MH}_n \) be arbitrary and suppose that Assumption 3.1 is satisfied by \( \tilde{f} \), that is

\[ \lim_{t \to \infty} \sum_{k=1}^{n-1} \tilde{f}_i^k(te_j + 1) = \infty, \quad i, j \in \{1, 2, \ldots, n\}. \]  

(5.3)

In this case assertions (i) and (ii) below are valid:

(i) If \( a > 0 \) is such that the lower eigenspace \( S_a(f) \) is nonempty, then \( S_a(f) \) is projectively bounded.

(ii) The second assertion in the statement of Theorem 4.1 holds.

\[ \lim_{t \to \infty} \sum_{k=1}^{n-1} \tilde{f}_i^k(te_j + 1) = \infty \iff \lim_{t \to 0} \bigwedge_{k=0}^{n-1} f_i^k(te_j + 1_{\{j\}}) = 0 \]

\[ \tilde{f}(y) = \mu y \] is equivalent to \( f(y^{-1}) = \mu^{-1} y^{-1} \),

\[ f \text{ has an eigenvalue } \iff \tilde{f} \text{ has an eigenvalue.} \]  

(5.4)
Example 5.1.

\[ h(x) = (a_1 x_1 \land a_2 x_2, \ b_1 x_1 \land b_2 x_2 \land b_3 x_3, \ c_1 x_1 \lor c_2 x_2)', \ x \in P_3, \quad (5.5) \]

\[ \tilde{h}(x) = (a_1^{-1} x_1 \lor a_2^{-1} x_2, \ b_1^{-1} x_1 \lor b_2 x_2 \lor b_3^{-1} x_3, \ c_1^{-1} x_1 \land c_2^{-1} x_2)', \ x \in P_3, \quad (5.6) \]

(a) \( \lim_{t \to \infty} \sum_{k=0}^{2} h_{k}^{k}([t - 1] e_1 + I) < \infty \)

Assumption 3.1 is not satisfied by \( h \), a conclusion that yields that \( M(h) \) is not a communicating matrix, by Theorem 3.1. Thus, the existence of an eigenvalue of \( h \) can not be obtained neither from Theorem 4.1 nor from Theorem 2.3.

(b) \[
M(\tilde{h}) = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

The existence of an eigenvalue of \( h \) can not be derived combining (5.4) with an application of Theorem 2.3 to the function \( \tilde{h} \).

(c) Assumption 3.1 is satisfied by \( \tilde{h} \). Thus, \( h \) has an eigenvalue, by Theorem 5.1. \( \Box \)
6. Problems

1. Necessary and Sufficient communication conditions so that \( S_a(f) \) ( \( S^a(f) \) ) is projectively bounded

2. Slice Spaces:
\[
S^b_a(f) := S_a(f) \cap S^b(f)
\]

Recession Function:
\[
\hat{f}(x) := \lim_{k \to \infty} (f(x^k))^{1/k}
\]
\[
\mathcal{E}(\hat{f}) = \{c1 | c > 0\} \Rightarrow S^b_a(f) \text{ is projectively bounded.}
\]

3. \( f_A(x) = Ax, A \) communicating:
\[
d_H(f^k_A(x), f^k_A(y)) \leq \beta^k d_H(x, y), \quad \beta \equiv \beta(A) < 1
\]
\[
f_A(x^*) = \lambda^* x^*
\]
\[
d_H(f^k_A(x), x^*) \leq \beta^k d_H(x, x^*) \to 0
\]
\[
\frac{f^{n+1}_A(x)}{f^n_A(x)} \to \lambda^* 1
\]

If a ‘general’ \( f \in \mathcal{M} \mathcal{H}_n \) has an eigenvector, and \( x \in \mathcal{P}_n \):

Analyze the behavior of \( \{d_H(f^k(x), \mathcal{E}(f))\} \) where \( \mathcal{E}(f) \) is the eigenspace of \( f \).