Particle Methods for Rare Event Monte Carlo

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March, 2009
The use of branching processes to estimate small probabilities
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- The design of such schemes was (until recently) poorly understood.
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Summary:

- The design of such schemes was (until recently) poorly understood.
- Design should be based on subsolutions to an associated HJB equation.
- Obtain necessary and sufficient conditions for asymptotically optimal performance.
Example: A tandem queue with server slow-down (Ethernet control)

$$\lambda \xrightarrow{} \begin{array}{c}
\mu_1/\nu_1
\end{array} \xrightarrow{} \begin{array}{c}
\mu_2
\end{array}$$

Also, analogous non-Markovian model.

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\[ \frac{\lambda}{\mu_1/\nu_1} \]

\[ Q_2 \]

\[ \begin{align*}
Q_1 & \quad d_1 \\
\theta & \quad \theta_n \\
\lambda & \quad \nu_3 \\
\mu_1 & \quad \mu_2
\end{align*} \]

\[ \begin{align*}
x_2 & \quad \theta_s \\
1 & \quad \theta_s \\
d_1 & \quad \theta
\end{align*} \]

\[ \begin{align*}
S & \quad D \\
d_2 & \quad d_1
\end{align*} \]
Example: A tandem queue with server slow-down (Ethernet control)

\[ p_n = P \{ Q_2 \text{ exceeds } n \text{ before } Q = (0, 0) \mid Q(0) = (1, 0) \} . \]
Example: A tandem queue with server slow-down (Ethernet control)

\[ p_n = P \{ Q_2 \text{ exceeds } n \text{ before } Q = (0,0) | Q(0) = (1,0) \} . \]

Also, analogous non-Markovian model.
As a general Markov model one can consider iid random vector fields \( \{ v_i(y), y \in \mathbb{R}^d \} \), and the process

\[
X_{i+1}^n = X_i^n + \frac{1}{n} v_i(X_i^n), \quad X_0^n = x.
\]
Model problem and large deviation scaling

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Define

\[
H(y, \alpha) = \log E \exp \left\langle \alpha, v_i(y) \right\rangle, \quad L(y, \beta) = \sup_{\alpha \in \mathbb{R}^d} \left[ \left\langle \alpha, \beta \right\rangle - H(y, \alpha) \right]
\]

\[
X^n(i/n) = X_i^n, \quad \text{piecewise linear interpolation for } t \neq i/n.
\]
Under conditions \( \{X^n(\cdot)\} \) satisfies a Large Deviation Principle with rate function

\[
I_T(\phi) = \int_0^T L(\phi, \dot{\phi}) dt
\]

if \( \phi \) is AC and \( \phi(0) = x \), and \( I_T(\phi) = \infty \) else.
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if \( \phi \) is AC and \( \phi(0) = x \), and \( I_T(\phi) = \infty \) else. Heuristically, for \( T < \infty \), given \( \phi \), small \( \delta > 0 \) and large \( n \)

\[
P \left\{ \sup_{0 \leq t \leq T} \|X^n(t) - \phi(t)\| \leq \delta \right\} \approx e^{-nI_T(\phi)}.
\]
Let $C = \{ \text{trajectories that hit } B \text{ prior to } A \}$. To estimate:

$$p_n(x) = P \{ X^n \in C | X^n(0) = x \}.$$
Under mild conditions:

\[-\frac{1}{n} \log p_n(x) \to \inf \{ I_T(\phi) : \phi \text{ enters } B \text{ prior to } A \text{ before } T, T < \infty \} = \gamma(x)\]
For standard Monte Carlo we average iid copies of $1\{X^n \in C\}$. One needs $K \approx e^{n\gamma}$ samples for bounded relative error $[\text{std dev} / p_n(x)]$. 

Performance determined by variance of $n_1$, and since unbiased by $E(n_1^2)$. 

By Jensen’s inequality $\frac{1}{n} \log E(n_1^2) \leq 2 \frac{1}{n} \log \frac{1}{p_n(x)}$. 

An estimator is called asymptotically efficient if $\lim inf n \rightarrow 1 \frac{1}{n} \log E(n_1^2) \leq 2 \frac{1}{n} \log \frac{1}{p_n(x)}$. 

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Some estimation generalities

1. For standard Monte Carlo we average iid copies of $1\{X^n \in C\}$. One needs $K \approx e^{n\gamma}$ samples for bounded relative error $[\text{std dev}/p_n(x)]$.

2. Alternative approach: construct iid random variables $\theta_1^n, \ldots, \theta_K^n$ with $E\theta_1^n = p_n(x)$ and use the unbiased estimator

$$\hat{q}_{n,K}(x) = \frac{\theta_1^n + \cdots + \theta_K^n}{K}.$$
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\(E(\theta_1^n)^2\).

4. By Jensen’s inequality

\[
-\frac{1}{n} \log E(\theta_1^n)^2 \leq -\frac{2}{n} \log E\theta_1^n = -\frac{2}{n} \log p_n(x) \to 2\gamma(x).
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3. Performance determined by variance of $\theta_1^n$, and since unbiased by $E(\theta_1^n)^2$.

4. By Jensen’s inequality

$$-\frac{1}{n} \log E(\theta_1^n)^2 \leq -\frac{2}{n} \log E\theta_1^n = -\frac{2}{n} \log p_n(x) \to 2\gamma(x).$$

5. An estimator is called *asymptotically efficient* if

$$\liminf_{n \to \infty} -\frac{1}{n} \log E(\theta_1^n)^2 \geq 2\gamma(x).$$
Pure branching methods (also called multi-level splitting)
Splitting type schemes

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- Branching with killing [RESTART, DPR]
Splitting type schemes

- Pure branching methods (also called multi-level splitting)
- Branching with killing [RESTART, DPR]
- Interacting particle systems (Del Moral et. al.)
Construction of splitting estimators

**Pure branching.** A certain number [proportional to $n$] of *splitting thresholds* $C_r^n$ are defined which enhance migration, e.g.,
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A single particle is started at \( x \) that follows the same law as \( X^n \), but branches into a number of independent copies each time a new level is reached.
The number of new particles $M$ can be random (though independent of past data), and a multiplicative weight $w_i$ is assigned to the $i$th descendant, where $E M X_i = 1$.
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$$E \sum_{i=1}^{M} w_i = 1.$$
Evolution continues until every particle has reached either $A$ or $B$. Let

\[
M^n_x = \text{total number of particles generated}
\]

\[
X^n_j(t) = \text{trajectory of } j\text{th particle},
\]

\[
W^n_j = \text{product of weights assigned to } j\text{ along path}
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Evolution continues until every particle has reached either $A$ or $B$. Let

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\[ X^n_j(t) = \text{trajectory of } j\text{th particle,} \]
\[ W^n_j = \text{product of weights assigned to } j \text{ along path} \]

Then

\[ \theta^n = \sum_{j=1}^{M^n_x} 1\{X^n_j \in C\} W^n_j \]
Subsolutions for branching processes

Now consider the asymptotic rate of decay as a function of $y$:

$$
\gamma(y) = \lim_{n \to \infty} \frac{1}{n} \log p_n(y) = \inf \{ I_T(\phi) : \phi(0) = y, \phi \text{ enters } B \text{ prior to } A \text{ before } T, T < \infty \}.
$$
Subsolutions for branching processes

Now consider the asymptotic rate of decay as a function of \( y \):

\[
\gamma(y) = \lim_{n \to \infty} - \frac{1}{n} \log p_n(y)
\]

\[
= \inf \{ I_T(\phi) : \phi(0) = y, \phi \text{ enters } B \text{ prior to } A \text{ before } T, T < \infty \}.
\]

Let

\[
\mathbb{H}(y, \alpha) = -H(y, -\alpha)
\]

[recall \( H(y, \alpha) = \log E \exp \langle \alpha, v_i(y) \rangle \)].
$\gamma(y)$ is a weak-sense solution to the PDE

$$\mathbb{H}(y, D\gamma(y)) = 0$$

$\gamma(y) = \infty$

$\gamma(y) = 0$
Subsolutions should satisfy (in the viscosity sense)

\[ \mathbb{M}(y, DW(y)) \geq 0 \]

\[ W(y) \leq 0 \]

\[ W(y) \leq \infty \]
Implementation and Performance for Pure Splitting [analogous results for RESTART, etc.]:

Consider a continuous function $W$ and suppose splitting levels are the level sets $f(W(y))_i \log EM = n$, where $EM$ is the mean number of particles per split. Then the number of particles needed to construct a single sample $n_1$ grows subexponentially if and only if $W$ is a viscosity subsolution. Given $u = EM$, consider particular scheme that randomizes between $b W_{bc}$ and $b W_{bc} + 1$ and uses weights $w_i = 1_u$. Then

$$\lim \inf_{n \to 1} n \log E(n_1^2 W(x) + x)^2.$$
Implementation and Performance for Pure Splitting [analogous results for RESTART, etc.]:
Consider a continuous function $W$ and suppose splitting levels are the level sets $\{W(y) \leq i \log EM/n\}$, where $EM$ is the mean number of particles per split.
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Given $u = EM$, consider particular scheme that randomizes between $\lfloor u \rfloor$ and $\lfloor u \rfloor + 1$ and uses weights $w_i = 1/u$. 
Statement of results

Implementation and Performance for Pure Splitting [analogous results for RESTART, etc.]:
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- Then the number of particles needed to construct a single sample $\theta_1^n$ grows subexponentially if and only if $W$ is a viscosity subsolution.

Given $u = EM$, consider particular scheme that randomizes between $\lfloor u \rfloor$ and $\lfloor u \rfloor + 1$ and uses weights $w_i = 1/u$. Then

- $\liminf_{n \to \infty} \frac{1}{n} \log E (\theta_1^n)^2 \geq W(x) + \gamma(x)$.
Remarks

- Subsolutions for interesting models (networks with feedback, non-Markovian systems, serve-the-longer discipline, server-slowdown dynamics, open/closed networks, path-dependent events) known.
- When available the Freidlin-Ventsel quasipotential can be used to construct subsolutions with optimal value.
- Subsolutions for importance sampling must be at least piecewise classical sense.
splitting for rare event simulation: A large deviations approach to
design and analysis (T. dean and D.), stochastic processes and their

A generalized DPR algorithm for rare event simulation (T. dean and
D.), submitted to annals of OR.

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