Recent Results in Large Population Mean Field Stochastic Dynamic Control Theory: Consensus Dynamics Derived from the NCE Equations

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Overview

- This research investigates:
  - Decision-making in stochastic dynamical systems with many competing agents
- Outline of contributions:
  - Nash Certainty Equivalence (NCE) Methodology
  - NCE for Linear-Quadratic-Gaussian (LQG) systems
  - Connection with physics of interacting particle (IP) systems
  - McK-V-HJB theory for fully nonlinear stochastic differential games
  - Invariance principle for controlled population behaviour
  - Models with interaction locality
  - Derivation the standard consensus dynamics from the NCE equations.

Some Facts and Implications

- Physics–Behavior of huge number of essentially identical infinitesimal interacting particles is basic to the formulation of statistical mechanics as founded by Boltzmann, Maxwell and Gibbs

- Game Theoretic Control System – Many competing agents
  - An ensemble of essentially identical players seeking individual interest
  - Individual mass interaction
  - Fundamental issue: how to relate individual actions to mass behavior?

Part I – Individual Dynamics and Costs

Individual dynamics:

$$dz_i = (a_i z_i + b u_i)dt + \alpha z^{(n)} dt + \sigma_i dw_i, \quad 1 \le i \le n. \quad (1$$

- z_i: state of the ith agent
- $z^{(n)}$: the population mean $z^{(n)} \triangleq \frac{1}{n}\sum_{i=1}^{n} z_i$
- u_i: control
- w_i: noise (a standard Wiener process)
- n: population size

For simplicity: Take the same control gain b for all agents.

Part I – Individ. Dynamics and Costs (ctn)

Individual costs:

$$J_i(u_i, \nu_i) = E \int_0^\infty e^{-\rho t}[(z_i - \nu_i)^2 + r u_i^2] dt \tag{2}$$

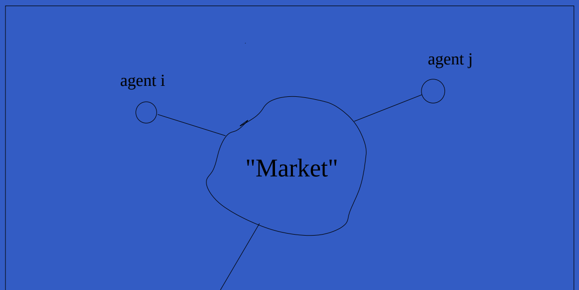
We are interested in the case $\nu_i = \Phi(z^{(n)}) \triangleq \Phi(\frac{1}{n}\sum_{k=1}^n z_k)$
Φ: nonlinear and Lipschitz

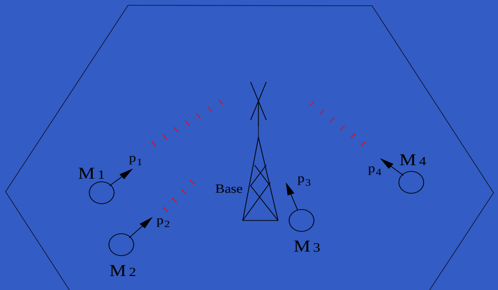
Main feature and Objective:

- Weak coupling via costs and dynamics
- Connection with IP Systems (for model reduction in McKean-Vlasov setting) with be clear later on
- Develop decentralized optimization

Part I – Motivational Background and Related Works

- Economic models (e.g., production output planning) where each agent receives average effect of others via Market (Lambson)
- Advertising competition game models (Erikson)
- Wireless network resource allocation (e.g., power control, HCM)
- Stochastic swarming (Morale et. al.); “selfish herd" (such as fish) reducing indiv. predation risk by joining group (Reluga & Viscido)
- Public health – Voluntary vaccination games (Bauch & Earn)
- Industry dynamics with many firms (Weintraub, Benkard, & Roy)
- Mathematical physics and finance (Lasry and Lions)
- Admission control in communication networks.





Part I – Motiv. Backgrd: Wireless Power Control

- Lognormal channel attenuation (in dB):

$$dx_i = -a(x_i + b)dt + \sigma dw_i, \qquad 1 \le i \le n.$$

- Additive power adjustment: $dp_i = u_i dt$.
- Individual Control Performance

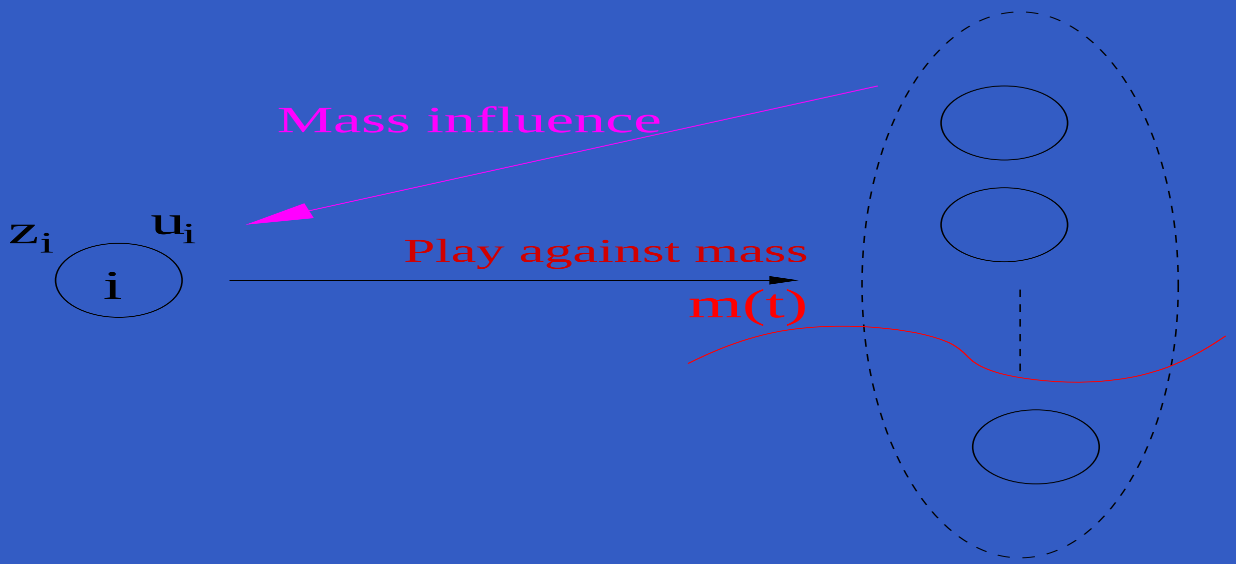
$$E \int_0^T \left\{ [e^{x_i} p_i - \alpha(\tfrac{\beta}{n} \textstyle\sum_{j=1}^n e^{x_j} p_j + \eta)]^2 + r u_i^2 \right\} dt.$$

  - The factor $\frac{\beta}{n}$ is due to linear increase of length of CDMA spreading seqnc w.r.t. user number. η: background noise.
  - Want matched filter output signal-to-interference ratio

$$\mathrm{SIR}_{output} = e^{x_i} p_i / \left[(\beta/n) \textstyle\sum_{j=1}^n e^{x_j} p_j + \eta \right]$$

    to stay near a certain target level. e^{x_i}: power attenuation from user to base.

Part I – Control Synthesis via NCE



- Under large population conditions, the mass effect concentrates into a deterministic quantity $m(t)$.
- A given agent only reacts to the mass effect $m(t)$ and any other individual agent becomes invisible.
- Key issue is the specification of $m(t)$ and associated individual action - Look for certain consistency relationships

Part II – Preliminary Optimal LQG Tracking

Take $f, z^* \in C_b[0,\infty)$ (bounded continuous) for scalar model:

$$\begin{aligned} d\hat{z}_i &= a_i \hat{z}_i dt + b u_i dt + \alpha f dt + \sigma_i dw_i \\ J_i(u_i, z^*) &= E \int_0^\infty e^{-\rho t}[(\hat{z}_i - z^*)^2 + r u_i^2] dt \end{aligned}$$

Riccati Equation : $\rho \Pi_i = 2a_i \Pi_i - \frac{b^2}{r}\Pi_i^2 + 1, \qquad \Pi_i > 0.$

Set $\beta_1 = -a_i + \frac{b^2}{r}\Pi_i$, $\beta_2 = -a_i + \frac{b^2}{r}\Pi_i + \rho$, and assume $\beta_1 > 0$.

Optimal Tracking Control $\longrightarrow \quad \widehat{u}_i = -\frac{b}{r}(\Pi_i z_i + s_i)$

Tracking Offset Equation $\longrightarrow \quad \rho s_i = \frac{ds_i}{dt} + a_i s_i - \frac{b^2}{r}\Pi_i s_i + \alpha \Pi_i f - z^*$

- Boundedness conditions uniquely determine s_i.

Part II – Notation

Based on LQ Riccati equation, denote:

$$\Pi_a = (\frac{b^2}{r})^{-1}\left[a - \frac{\rho}{2} + \sqrt{(a-\frac{\rho}{2})^2 + \frac{b^2}{r}}\right],$$

$$\beta_1(a) = -\frac{\rho}{2} + \sqrt{(a-\frac{\rho}{2})^2 + \frac{b^2}{r}}, \qquad (3$$

$$\beta_2(a) = \frac{\rho}{2} + \sqrt{(a-\frac{\rho}{2})^2 + \frac{b^2}{r}}. \qquad (4$$

$$\Longrightarrow \Pi_a = (\frac{b^2}{r})^{-1}(a + \beta_1(a)).$$

Part III – Population Parameter Distribution

Define empirical distribution associated with first n agents

$$F_n(x) = \frac{\sum_{i=1}^{n} 1_{(a_i<x)}}{n}, \qquad x \in \mathbb{R}.$$

- (**H1**) There exists a distribution F s.t. $F_n \to F$ weakly.

- Each agent is given its “a" parameter which it knows.

- Information on Other Agents is available statistically in terms of the empirical distribution. Specifically, assume F is known.

Part III – LQG-NCE Equation Scheme

Assume zero initial mean, i.e., $Ez_i(0) = 0$, $i \geq 1$. Based on population limit, the Fundamental NCE equation system:

$$\rho s_a = \frac{ds_a}{dt} + a s_a - \frac{b^2}{r}\Pi_a s_a + \alpha \Pi_a \bar{z} - z^*, \quad (5$$

$$\frac{d\overline{z}_a}{dt} = (a - \frac{b^2}{r}\Pi_a)\overline{z}_a - \frac{b^2}{r} s_a + \alpha \bar{z}, \quad (6$$

$$\overline{z} = \int_{\mathcal{A}} \overline{z}_a dF(a), \quad (7$$

$$z^* = \Phi(\overline{z}). \quad (8$$

Basic idea behind NCE(z^*) with parameters $F(\cdot)$, a, b, α, r:

- Solve z^* tracking problem for one agent.
- Use popul. average $\overline{z}$ to approximate coupling term $\frac{1}{n}\sum_k^n z_k$.
- Individual action u_i is optimal response to z^*.
- Collectively produce same z^* assumed in first place.

Part IV – Summary of NCE for LQG Model

Recall the system of n agents with dynamics:

$$dz_i = a_i z_i dt + b u_i dt + \alpha z^{(n)} dt + \sigma_i dw_i, \qquad 1 \le i \le n, \qquad t \ge 0.$$

Let u_{-i} denote the row $(u_1, \cdots, u_n)$ with u_i deleted, and reexpress the individual cost

$$J_i(u_i, u_{-i}) \triangleq E \int_0^\infty e^{-\rho t} \{[z_i - \Phi(\frac{1}{n}\sum_{k=1}^n z_k)]^2 + r u_i^2\} dt.$$

Denote the optimal control for the tracking problem with s_i pre-computed from the deterministic LQG NCE by

$$u_i^0 = -\frac{b}{r}(\Pi_i z_i + s_i), \quad 1 \le i \le n,$$

revealing the closed-loop fixed point form of the large population tracking problem!

Part IV – Main Existence Results

Theorem **(Existence and Uniqueness)** The NCE(z^*) equation system has a unique bounded solution $(\bar{z}_a, s_a)$ for each $a \in \mathcal{A}$ subject to **(H2)**-**(H3)**. □

(H1) There exists a distribution F s.t. $F_n \to F$ weakly. (restated)

(H2) Φ is Lipschitz with parameter γ.

(H3) Gain condition: $\int_{\mathcal{A}} \left[\frac{|\alpha|}{\beta_1(a)} + \frac{b^2(\gamma+|\alpha|\Pi_a)}{r\beta_1(a)\beta_2(a)} \right] dF(a) < 1$, and $\beta_1(a) > 0$ for all $a \in \mathcal{A}$.

(H4) All agents have independent initial conditions with zero mean, and $\sup_{i \geq 1}[\sigma_i^2 + E z_i^2(0)] < \infty$.

Part IV – Asymptotic Equilibrium

The k-th agent's admissible control set $\mathcal{U}_k$ consists of all feedback controls u_k adapted to $\sigma(z_i(\tau), \tau \le t, 1 \le i \le n)$.

Definition A set of controls $u_k \in \mathcal{U}_k, 1 \le k \le n$, for n players is called an *ε-Nash equilibrium* w.r.t. the costs $J_k, 1 \le k \le n$, if there exists $\varepsilon \ge 0$ such that for any fixed $1 \le i \le n$, we have

$$J_i(u_i, u_{-i}) \le J_i(u_i', u_{-i}) + \varepsilon,$$

when any alternative $u_i' \in \mathcal{U}_i$ is applied by the i-th player. $\square$

Part IV – Stability and Equilibria

Theorem The set of controls $\{u_i^0, 1 \le i \le n\}$ results in second order stability & an ε-Nash equilibrm w.r.t. costs $J_i(u_i, u_{-i})$, $1 \le i \le n$, i.e.,

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \le \inf_{u_i} J_i(u_i, u_{-i}^0) \le J_i(u_i^0, u_{-i}^0)$$

where $0 < \varepsilon \to 0$ as $n \to \infty$, and $u_i \in \mathcal{U}_i$ is any alternative control which depends on $(t, z_1, \cdots, z_n)$, and

$$u_i^0 = -\tfrac{b}{r}(\Pi_i z_i + s_i). \qquad \square$$

- For uniform agents, $\varepsilon = O(1/\sqrt{n})$.
- For non-uniform agents, the bound estimates depend on limiting behavior of $F_n \to F$ (weakly).
- Performance analysis: approximating $(z_1, \cdots, z_n)$ in closed-loop by n independent copies of the McK-V equation driven by $(z_i(0), w_i)$ associated with z_i.

Part IV – Implication for Rational Expectations

- Rational Expectations in Macroeconomic Theory. Issue of how economic agents forecast future events (and hence play against macroeconomic policy)

- NCE theory gives a coherent and tractable formulation of Rational Expectations in game theoretic economic behaviour with a large number of players. Implications for macroeconomic policy?

- In particular, NCE provides a means for maintaining RE in that by this mechanism each individual can forecast
  - the overall population behaviour, and
  - the associated optimal individual responses

Part IV – Explicit Solutions for LQG-NCE Equation System

For a system of uniform agents with $a_i = a$, $\Phi(z) = \hat{\gamma}(z+\eta)$.

$$\text{NCE} \Longrightarrow \quad \begin{aligned} \rho s &= \frac{ds}{dt} + as - \frac{b^2}{r}\Pi s + \alpha \Pi \bar{z} - z^*, \\ \frac{d\bar{z}}{dt} &= (a - \frac{b^2}{r}\Pi)\bar{z} + \alpha \bar{z} - \frac{b^2}{r} s, \\ z^* &= \phi(\bar{z}) = \hat{\gamma}(\bar{z} + \eta). \end{aligned}$$

$\Downarrow$ (steady-state)

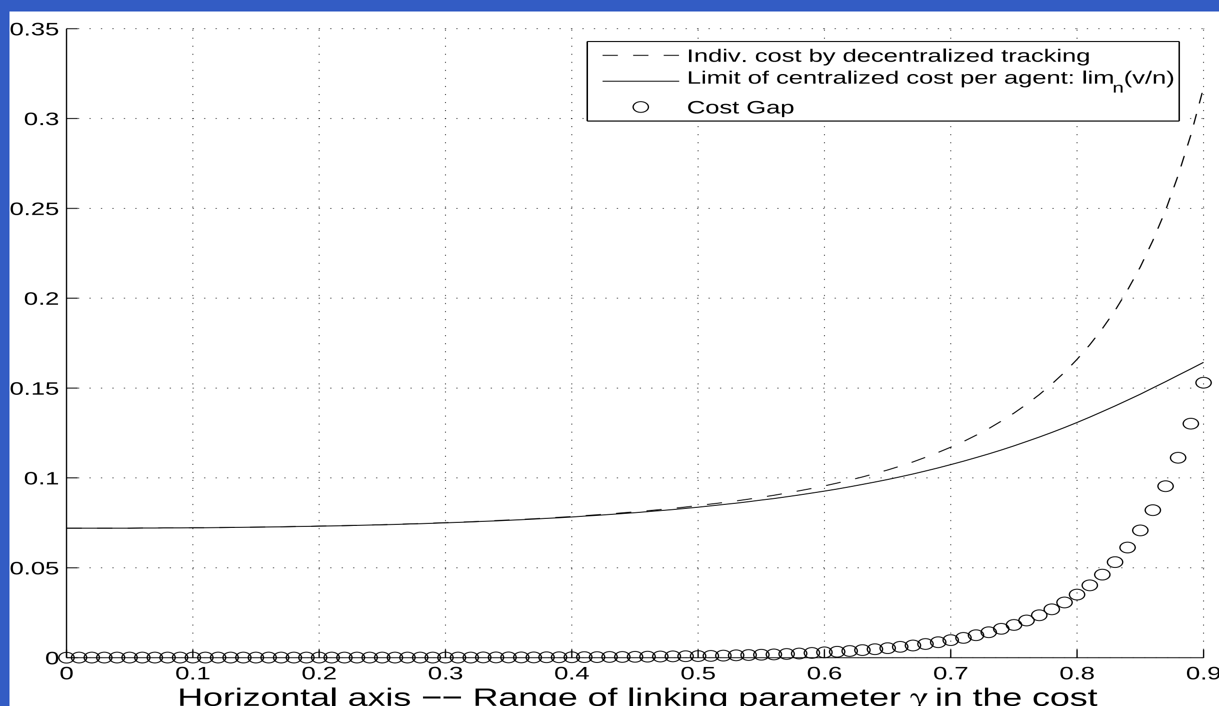
$$\begin{cases} \beta_2 s(\infty) - \alpha \Pi \bar{z}(\infty) + z^*(\infty) = 0 \\ -\frac{b^2}{r} s(\infty) + (\alpha - \beta_1)\bar{z}(\infty) = 0 \\ \hat{\gamma}\bar{z}(\infty) - z^*(\infty) = -\hat{\gamma}\eta. \end{cases}$$

$\Downarrow$

unique solution

Part IV – Cost Gap

- Solve an LQG game model involving cost coupling with individual cost J_i. Denote Nash equilibrium cost v_{ind} with population limit.
- Take welfare function $J = \sum_{i=1}^{n} J_i$ and compute optimal control with cost v_n. Optimal centralized control cost per agent $\bar{v} = \lim_{n\to\infty} v_n/n$.
- Cost gap: $v_{ind} - \bar{v}$.



Part V – Fully Nonlinear Models and McK-V-HJB Approach

- Dynamics:

$$dz_i = (1/n)\sum_{j=1}^{n} f(z_i, u_i, z_j)dt + \sigma dw_i, \quad 1 \le i \le n, \quad t \ge 0,$$

- Costs:

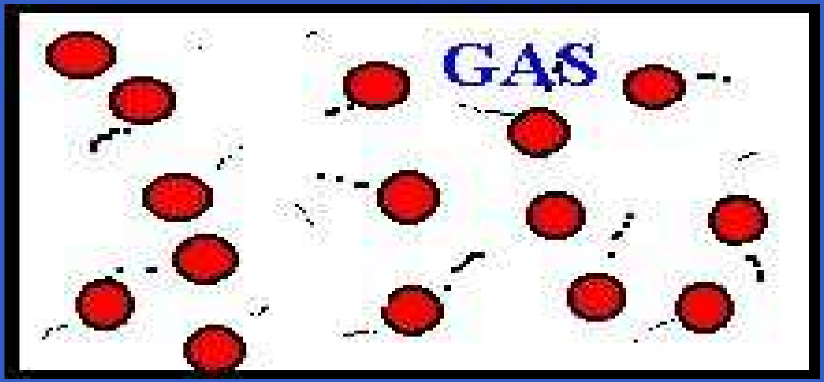
$$J_i(u_i) \triangleq E\int_0^T \Big[(1/n)\sum_{j=1}^{n} L(z_i, u_i, z_j)\Big]dt, \quad T < \infty.$$

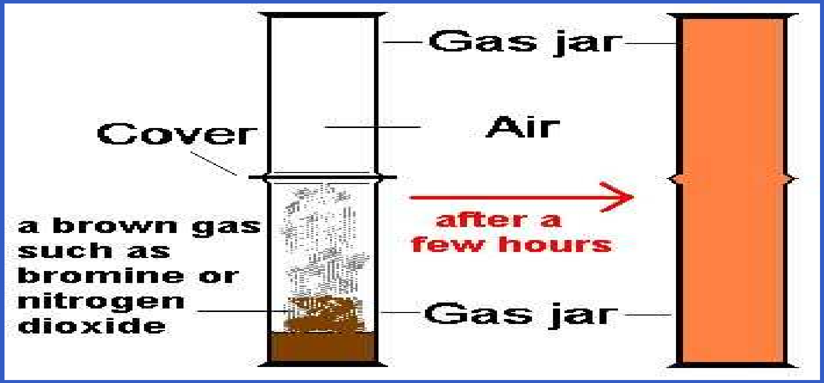
- Control set: Each $u_k \in U$ compact.

- Other variants of the cost may be considered.

- Objective: look for decentralized strategies

Part V – Connection with Statistical Mechanics





- Boltzmann PDE describing evolution of spatial-velocity ($x - v$) distribution $u(t, x, v)$ of huge number of gas particles
- Solution to spatially homogeneous Boltzman PDE (for $u(t, v)$) has a probabilistic interpret. via McKean's Markov system:
  - Generator depends on "current density" of the process
  - Thus, there exists a driving effect from the mass
    - This feature also appears in our diffusion based models, where current density affects the drift

Part V – Controlled McKean-Vlasov Equations

- Controlled McK-V equation via a representative agent:

$$dx_t = f[x_t, u_t, \mu_t]dt + \sigma dw_t,$$

where $f[x, u, \mu_t] = \int_{\mathbb{R}} f(x, u, y)\mu_t(dy)$.

- Individual cost:

$$J(u, \mu) \triangleq E \int_0^T L[x_t, u_t, \mu_t]dt,$$

where $L[x, u, \mu_t] = \int_{\mathbb{R}} L(x, u, y)\mu_t(dy)$.

- Generalization to multi-class agents corresponds to non-uniform agent case in basic NCE analysis.

Part V – The McK-V-NCE Principle

- Methodology: The key steps are to construct a mutually consistent pair of
  - (i) the mass effect, and
  - (ii) the individual strategies such that the latter not only
    - (a) each constitute an optimal response to the mass effect
    - (b) but also collectively produce that mass effect.
  - In non-uniform NCE-McKV setting, the mass effect is an average w.r.t. the agent type distribution F_a.

- Principle: The application of an appropriate, general, Fixed Point Theorem demonstrates that such a solution
  - exists, is unique
  - and is collectively produced by the actions of the individual agents.

Part V – NCE and McK-V-HJB Theory

- HJB equation:

$$-\frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x,u,\mu_t]\frac{\partial V}{\partial x} + L[x,u,\mu_t] \right\} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}$$
$$V(T,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R}.$$

$$\Downarrow$$

$$\text{Optimal Control}: \quad u_t = \varphi(t, x|\mu_\cdot), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

- Closed-loop McK-V equation:

$$dx_t = f[x_t, \varphi(t,x|\mu_\cdot), \mu_t]dt + \sigma dw_t, \quad 0 \le t \le T.$$

The NCE methodology amounts to finding a solution (x_t, μ_t) in McK-V sense.

Part V –Outline of Analysis Based on NCE

By the NCE methodology, we carry out the steps:

- Construct controlled McKean-Vlasov equation; fixed point theory for existence analysis

- Develop HJB equation (involving a measure flow) and derive Optimal Response Mapping for individuals

- Establish existence results (for McK-V-HJB system)

- For equilibrium analysis – Approximate n “controlled interacting particles" in closed-loop by n independent copies of the McK-V equation

Part VI – Asymptotic Nash Equilibrium

Theorem (Individual Level – Nash) Under mild conditions, the set of McK-V-HJB based controls $\{u_i^0, 1 \le i \le N\}$ results in an ε-Nash equilibrium w.r.t. costs $J_i(u_i, u_{-i})$, $1 \le i \le N$, i.e.,

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \le \inf_{u_i} J_i(u_i, u_{-i}^0) \le J_i(u_i^0, u_{-i}^0)$$

where $0 < \varepsilon \to 0$ as $N \to \infty$.

Part VI – Generalization to Multi-class Agents

- Dynamics:

$$dz_i = (1/n) \sum_{j=1}^{n} f_{a_i}(z_i, u_i, z_j)dt + \sigma dw_i, \quad 1 \le i \le n, \quad t \ge 0,$$

  where a_i is the dynamic parameter, indicating type of agent.

- Costs:

$$J_i(u_i) \triangleq E \int_0^T \Big[(1/n) \sum_{j=1}^{n} L(z_i, u_i, z_j) \Big] dt, \quad T < \infty.$$

- Control set: Each $u_k \in U$.

- The sequence $\{a_i, i \ge 1\}$ takes value from $\mathcal{A} = \{\theta_1, \cdots, \theta_K\}$ with empir. distri. $(\pi_1, \cdots, \pi_K)$, i.e., $(1/n) \sum_{i=1}^{n} 1_{(a_i = \theta_k)} \to \pi_k$.

Part VI – Generalizat'n to Multi-class Agents (ctn)

- Controlled McK-V equation via a representative agent:

$$dx_t = f_a[x_t, u_t, \mu_t^1, \cdots, \mu_t^K]dt + \sigma dw_t,$$

where $f_a[x, u, \mu_t^1, \cdots, \mu_t^K] = \sum_{k=1}^K \pi_k \int_{\mathbb{R}} f_a(x, u, y)\mu_t^k(dy)$.

- μ_t^k reproduces the mass interaction generated by the class of agents with parameter $a = \theta_k$. $(\pi_1, \cdots, \pi_K)$: para empir. distri.

- Individual cost: $J(u, \mu) \triangleq E \int_0^T L[x_t, u_t, \mu_t^1, \cdots, \mu_t^K]dt$, where $L[x, u, \mu_t^1, \cdots, \mu_t^K] = \sum_{k=1}^K \pi_k \int_{\mathbb{R}} L(x, u, y)\mu_t(dy)$.

- Distribution over agents would give generalized McK-V-HJB with integral over the agent measures on right hand side.

Along the optimal controlled trajectory, let

$$\xi_t \triangleq \int_0^t L(z(s), u^*(s, z(s)))ds + V(t, z(t))$$

where $t \in [0, T]$. In stochastic optimal control, it is well known that ξ_t is a martingale.

$z(t)$: closed-loop solution when optimal control u^* applied.

$V(t, z(t))$: value function associated with $(t, z(t))$.

- In the game problem, each agent essentially solves a local optimal control problem.
- Implication for the large population game when the Nash strategies are collectively applied?

Part VII – The Invariance Principle (ctn)

For the McKean-Vlasov-HJB equation, we make existence assumptions:

- (A1) There exists a solution $(x_i(t), V_{a_i}(t, x_i), \hat{u}_i(t, x_i))$ to the McKean-Vlasov-HJB system for multi-class agents.
- (A2) The closed-loop drift coefficient $f_{a_i}(x_i, \hat{u}_i(t, x_i))$ is in $C([0, T] \times \mathbb{R})$ and Lipschitz continuous in x_i.
- (A3) Under the control $\hat{u}_i$, $L[x_i, \hat{u}_i(t, x_i), \mu_t^o]$ is in $C([0, T] \times \mathbb{R})$ and has a polynomial growth rate with respect to x_i.

We denote $\mu_t^o = [\mu_t^1, \cdots, \mu_t^K]$.

Part VII – The Invariance Principle (ctn)

We use x_i, $i = 1, 2, 3 \cdots$, to denote a sequence of copies of processes generated by the McKean-Vlasov equation.

Theorem Suppose (A1)-(A3) hold. Then the process

$$\int_0^t L[x_i(s), \hat{u}_i(s, x_i(s)), \mu_s^o] ds + V_{a_i}(t, x_i(t))$$

is a martingale.

By averaging across of the population limit, we get a deterministic martingale, hence a constant:

$$c = \int_0^t \int_{\mathbb{R}^2} \sum_{i,j=1}^{K} \pi_i \pi_j L(x, \hat{u}_{\theta_i}(s, x), y) \mu_s^j(dy) \mu_s^i(dx) ds + \int_{\mathbb{R}} \sum_{i=1}^{K} \pi_i V_{\theta_i}(t, x) \mu_t^i(dx)$$

c: determined by the initial condition of the population.

θ_i: indicates the type of the agent.

After applying the McK-V-HJB based control laws to the population of N agents, we can further show

Theorem (Large Population Invariance Princple)

$$\varepsilon_N \triangleq \left| \frac{1}{N} \sum_{i=1}^{N} \int_0^t L[z_i(s), \hat{u}_i(s, z_i(s)), \mu_s^o] ds + \frac{1}{N} \sum_{i=1}^{N} V_{a_i}(t, z_i(t)) - c \right|$$

tends to zero in L_2, as $N \to \infty$.

Part VII – Computational Example for LQG Systems

Resulting from the invariance principle:

$$c = \sum_{k=1}^{K} \pi_k \Big\{ \int_0^t \int_{\mathbb{R}} \Big[(x - z^*(\tau))^2 + \frac{b^2}{r}(\Pi_{\theta_k}(\tau)x + s_{\theta_k}(\tau))^2\Big] dF^\tau_{\theta_k}(x)d\tau$$

$$+ \int_{\mathbb{R}} [x^2\Pi_{\theta_k}(t) + 2xs_{\theta_k}(t)]dF^t_{\theta_k}(x) + q_{\theta_k}(t)\Big\}.$$

$F^\tau_{\theta_k}(x)$: state distri. at τ for an agent with dynamic parameter θ_k.
Assume existence of density $p^t_{\theta_k}(x)$ with suitable regularity; then by taking differentiation, we get

$$0 = \sum_{k=1}^{K} \pi_k \Big\{ \int_{\mathbb{R}} \Big[(x - z^*(t))^2 + \frac{b^2}{r}(\Pi_{\theta_k}(t)x + s_{\theta_k}(\tau))^2\Big] p^t_{\theta_k}(x)dx$$

$$+ \int_{\mathbb{R}} \frac{\partial[x^2\Pi_{\theta_k}(t) + 2xs_{\theta_k}(t)]p^t_{\theta_k}(x)}{\partial t} dx + \frac{dq_{\theta_k}(t)}{dt} \Big\}.$$

Part VIII — Generalization with Interaction Locality

- Related Background:
  - Social segregation (Schelling, 1971); 1-D line topology
  - Retailing services (Blume, 1993); 2-D lattice topology
- The individual dynamics:

$$dz_i(t) = [az_i(t) + bu_i(t)]dt + \sigma dW_i(t), \quad 1 \le i \le N, \quad t \ge 0,$$

- The cost with interaction locality:

$$J_i = E\int_0^\infty e^{-\rho t}\left\{[z_i - \tilde{\Phi}_i]^2 + ru_i^2\right\}dt,$$

where $\tilde{\Phi}_i = \gamma(\sum_{j=1}^N \omega_{p_i p_j}^{(N)} z_j + \eta)$ and $\rho > 0$, $\gamma > 0$, $r > 0$.

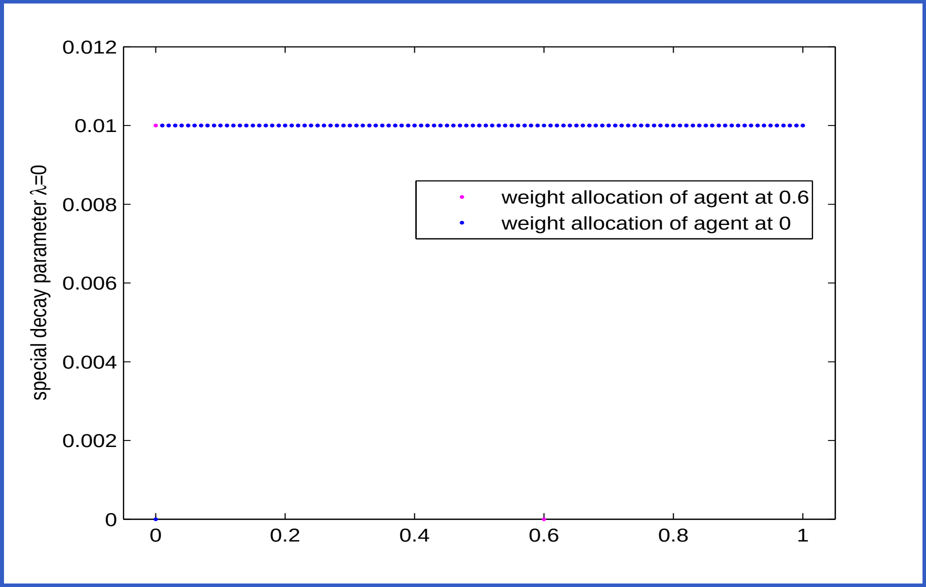
- Weight allocation — The set of weight coefficients $\omega_{p_i p_j}^{(N)}$ satisfies

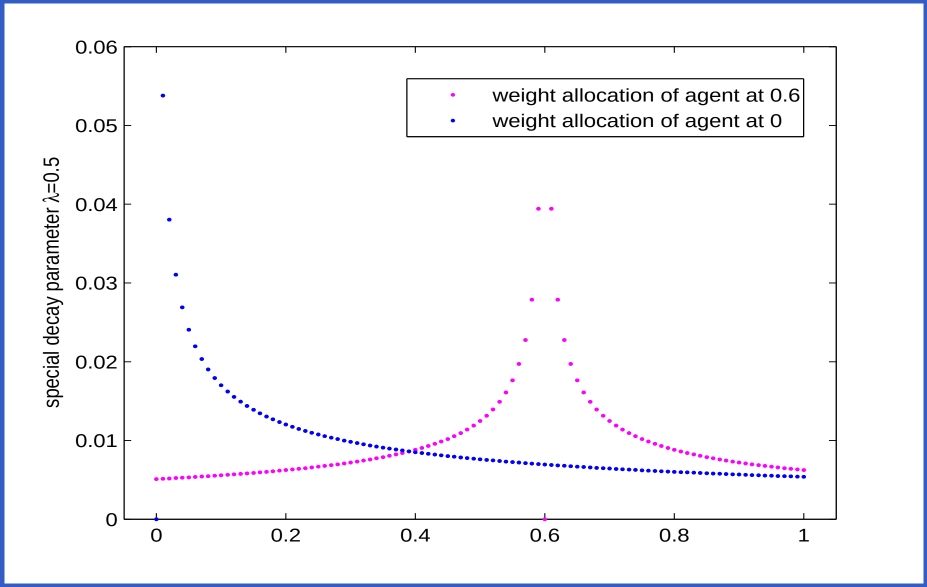
$$\omega_{p_i p_j}^{(N)} \ge 0, \quad \forall i, j, \qquad \sum_{j=1}^N \omega_{p_i p_j}^{(N)} = 1, \quad \forall i.$$

Part VIII — Example of Weight Allocation

For illustration, consider the 1-D interaction:

- Partition $[0, 1]$ with stepsize 0.01 to get $N = 101$ locations
- Label the N locations consecutively by $p_1, \cdots, p_N$.
- Let $\omega^{(N)}_{p_i p_j} = c|p_i - p_j|^{-\lambda}$ where $\lambda \in [0, 1]$ and c is normalizing factor so that all weights add up to one.





(a) uniform/flat allocation

$\lambda = 0$

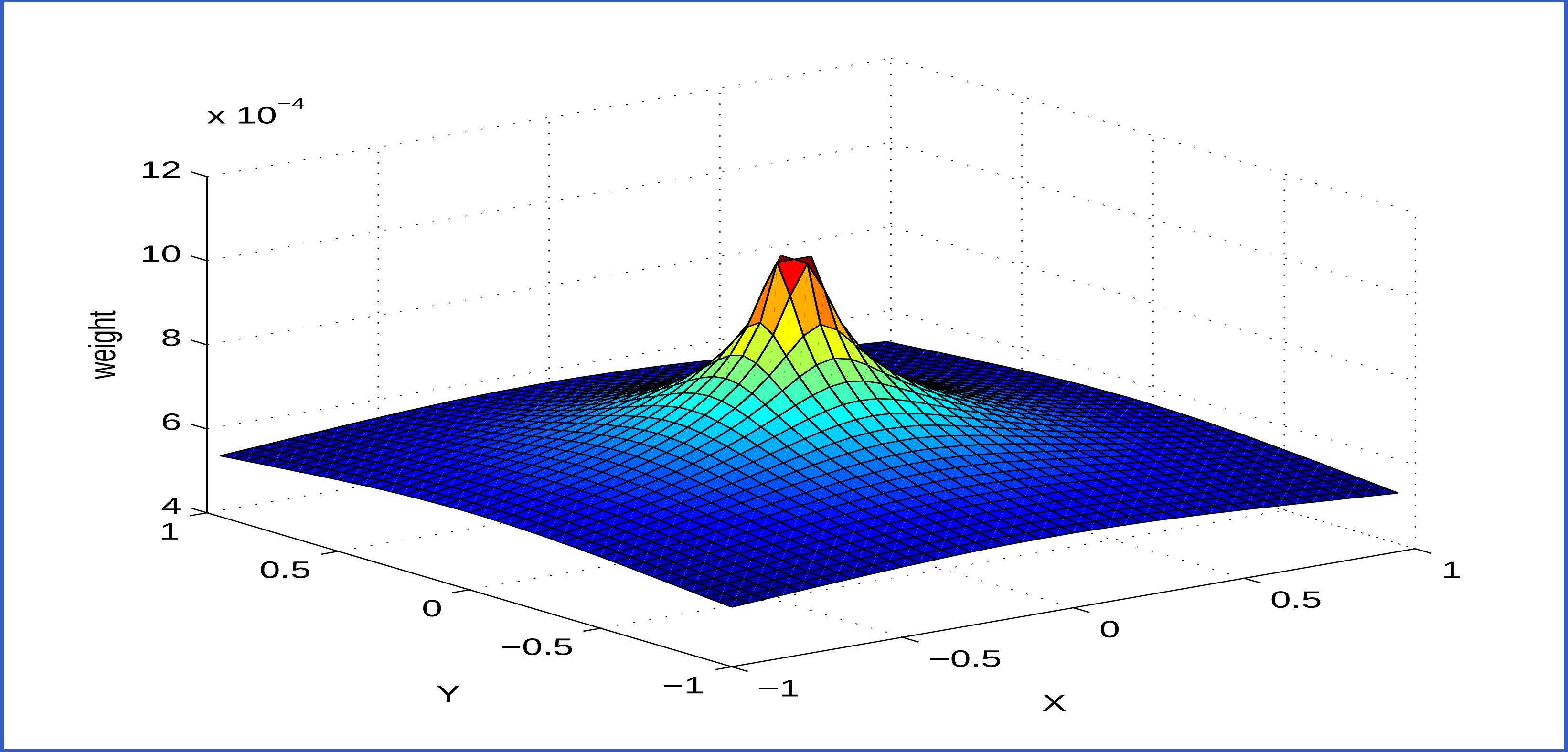
(b) distance-dependent allocation

$\lambda = 0.5$

Part VIII — Example of Weight Allocation (ctn)

Consider the 2-D interaction

- Partition $[-1, 1] \times [-1, 1]$ into a 2-D lattice
- Weight decays with distance by the rule $\omega_{p_i p_j}^{(N)} = c|p_i - p_j|^{-\lambda}$ where c is the normalizing factor and $\lambda \in [0, 2]$



Part VIII — Notation and Assumptions

Again, let $\Pi_a > 0$ be the solution to the algebraic Riccati equation:

$$\rho\Pi = 2a\Pi - \frac{b^2}{r}\Pi^2 + 1. \quad (9$$

Denote $\beta_1 = -a + \frac{b^2}{r}\Pi_a$ and $\beta_2 = -a + \frac{b^2}{r}\Pi_a + \rho$.

Take $[\underline{\alpha}, \overline{\alpha}]$ as the locality index set (i.e., line topology)

(C1) $F_\alpha(\alpha')$: $[\underline{\alpha}, \overline{\alpha}] \times \mathbb{R} \rightarrow [0, 1]$ satisfies: i) $F_\alpha(\cdot)$ is a probab. distrib. function $\forall\, \alpha$, $\int_{\alpha' \in [\underline{\alpha}, \overline{\alpha}]} dF_\alpha(\alpha') = 1$; ii) $\int_{\alpha' \in B} dF_\alpha(\alpha')$ is a measurable function of α for each Borel subset B of $\mathbb{R}$; iii) $F_{\alpha''}(\cdot)$ converges to $F_\alpha(\cdot)$ weakly when $\alpha'' \rightarrow \alpha$, where α and α'' are in $[\underline{\alpha}, \overline{\alpha}]$. □

(C2) The constants $\beta_1 > 0$, $\beta_2 > 0$, and the ratio $(\gamma b^2)/(r\beta_1\beta_2) < 1$. □

Part VIII — NCE Equation with Interaction Locality

The Localized NCE (Mean Field) equation system:

$$\rho s_\alpha = \frac{ds_\alpha}{dt} + a s_\alpha - \frac{b^2}{r}\Pi_a s_\alpha - R_\alpha, \quad (10$$

$$\frac{d\bar{z}_\alpha}{dt} = (a - \frac{b^2}{r}\Pi_a)\bar{z}_\alpha - \frac{b^2}{r}s_\alpha, \quad (11$$

$$\bar{r}_\alpha(t) = \int_{\alpha' \in [\underline{\alpha}, \overline{\alpha}]} \overline{z}_{\alpha'}(t) dF_\alpha(\alpha'), \quad (12$$

$$R_\alpha = \gamma(\bar{r}_\alpha + \eta). \quad (13$$

- Remark: The mean field effect now depends on the location of the agent in question

Theorem Under (**C1**)-(**C2**), there exists a unique bounded solution $(s_\alpha(\cdot), \bar{z}_\alpha(\cdot), r_\alpha(\cdot))$ to the NCE equation system (10)-(13).

Part VIII — Assumptions on Weight Allocation

(**C3**) The weight allocation satisfies the condition

$$\epsilon_N^{\omega} \triangleq \sup_{1\le i\le N} \sum_{j=1}^{N} |\omega_{p_i p_j}^{(N)}|^2 \to 0, \quad \text{as} \quad N \to \infty. \qquad \square$$

- Roughly, this condition implies the weight cannot highly concentrate on a small number of neighbors; if the decay rate $\lambda \in [0,1]$, (**c3**) holds
- When the decay rate $\lambda > 1$, (**c3**) and then deterministic mean field approximation fail

(**C4**) For each p_i, the empirical distribution

$$F_{p_i}^{(N)}(x) = \sum_{p_j < x} \omega_{p_i p_j}^{(N)}, \qquad x \in \mathbb{R},$$

is associated with a distribution function $F_{p_i}(x)$ (specified in (**c1**)) such that for any $\delta > 0$, there exists a compact subset $D_{p_i}^N$ of $I = [\underline{\alpha}, \overline{\alpha}]$ with Lebesgue measure meas$(D_{p_i}^N) < \delta$, and

$\lim_{N\to\infty} \sup_{1\le i\le N} \sup_{x\in I\setminus D_{p_i}^N} |F_{p_i}^{(N)}(x) - F_{p_i}(x)| = 0.$ $\square$

Part VIII — Equilibrium Analysis

Theorem Under (**C1**)-(**C4**), given any $\varepsilon > 0$, there exists N_ε such that for all $N \geq N_\varepsilon$, the set of control strategies $\{\hat{u}_i, 1 \leq i \leq N\}$ is an ε-Nash equilibrium w.r.t. costs $J_i(u_i, u_{-i})$, $1 \leq i \leq N$, i.e.,

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \leq \inf_{u_i} J_i(u_i, u_{-i}^0) \leq J_i(u_i^0, u_{-i}^0), \quad (14$$

where

$$\hat{u}_i = -\frac{b}{r}(\Pi_a z_i + s_{p_i})$$

and s_{p_i} is given by the new NCE equation system (10)-(13) via the substitution $\alpha = p_i$ in s_α. □

Note: There is a further ramification of the main theorem:

- the population includes several classes of agents,
- and the interaction strength is specified according to inter/intra subpopulation interaction

Part IX – Consensus Problem : Background

- Consensus means both the agreement between agents of the group and the process of reaching to such an agreement.
- In standard consensus algorithms, there is a network of agents with dynamics:

$$\dot{z}_i(t) = u_i(t), \quad 1 \leq i \leq n \tag{15}$$

  interested in reaching an agreement via local communications with their neighbours on a graph $G = (\mathcal{V}, \mathcal{E})$.

- It is shown that the linear system

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(z_j(t) - z_i(t)), \tag{16}$$

  is a distributed consensus algorithm which guarantees agreement under suitable connectivity assumption.

Part IX – Consensus Problem : Background (cnt)

The dynamics of (16) can be stated in the vector form

$$\dot{z}(t) = -Lz(t), \tag{17}$$

where $z = (z_1, \cdots, z_n)^T$ is the state vector and L is the graph Laplacian:

$$L = D - A,$$

A is the adjacency matrix

$$[A]_{ij} = \begin{cases} a_{ij} & (j, i) \in \mathcal{E}, \\ 0 & \text{otherwise,} \end{cases}$$

and $D = diag(d_1, \cdots, d_n)$ is the degree matrix of G, $d_i = \sum_{j \neq i} a_{ij}$.

Part IX – Consensus Problem : Background (cnt)

Theorem

Consider a network of N agents on a graph $G(A)$, with dynamics

$$\dot{z}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(z_j(t) - z_i(t)). \quad (18$$

Suppose $G(A)$ is a strongly connected digraph and λ is a left eigenvector associated with a simple zero eigenvalue of $L(G)$, i.e. $\lambda^T L = 0$. Then

- a consensus is asymptotically reached for all initial states;
- the group consensus is $\alpha = \sum_i \mu_i z_i(0)$ where $\mu_i = \frac{\lambda_i}{\sum_i \lambda_i}$.
- if the digraph is balanced, i.e $\sum_{i \neq j} a_{ij} = \sum_{i \neq j} a_{ji}$ for all j, an average consensus, $\alpha = (\sum_i z_i(0))/N$, is asymptotically reached.

Part IX – Stochastic Consensus Problem by NCE

- We propose a new approach to consensus problem by using the NCE methodology.
- The stochastic dynamics for an individual agent is:

$$dz_i(t) = u_i(t)dt + CdW_i(t),\ t \geq 0,\ 1 \leq i \leq N, \tag{19}$$

  - $z_i \in \mathbb{R}^n$: the state of agent i,
  - $u_i \in \mathbb{R}^n$: control input,
  - $\{W_i, 1 \leq i \leq N\}$: independent d-D Wiener processes,
  - $C \in \mathbb{R}^{n \times d}$: the noise intensity matrix.

Part IX – Stochastic Consensus Problem by NCE (cnt)

- The general Long Range Average (LRA) LQG cost for agent i:

$$J_i \triangleq \limsup_{T\to\infty} \frac{1}{T}\int_0^T \{(z_i - \Phi_i)^T Q(z_i - \Phi_i) + u_i^T R u_i\} dt, \quad (20$$

  - $Q = Q^T \geq 0$, $R = R^T > 0$,
  - $\Phi_i = \gamma \sum_{j=1}^N \omega_{p_i p_j}^{(N)} z_j$, $\omega_{p_i p_j}^{(N)}$: the set of weight coefficients.
- Weight coefficient matrix $\Omega = (\omega_{p_i p_j}^N)$ is a normalized stochastic matrix.

Stochastic consensus dynamics (19)

+ LRA (localized) LQG costs (20)

$\Rightarrow$ Localized MF formulation of the Consensus Problem.

Part IX – Stochastic Consensus Problem by NCE (cnt)

- The NCE methodology for agents with LRA cost (20) and uniform weights, $\omega_{p_i p_j}^{(N)} = \frac{1}{N}$, has been studied in (Li, Zhang TAC08).

Definition A set of controls u_k, $1 \le k \le N$, is called an asymptotic Nash equilibrium in probability with respect to the costs J_k, if for any $\epsilon > 0$, $\delta > 0$ and fixed i, $1 \le i \le N$, there exist $N_{\epsilon,\delta}$ such that for any $N > N_{\epsilon,\delta}$

$$P\left(\sup_{1 \le i \le N} \left(J_i(u_i, u_{-i}) - \inf_{v_i} J_i(v_i, u_{-i}) \right) \ge \delta \right) \le \epsilon.$$

- Li, Zhang has shown that the decentralized control laws have the asymptotic Nash-equilibrium property in the probabilistic sense.

Part IX – Stochastic Consensus Problem by NCE (cnt)

NCE equations of the localized MF formulation of the consensus problem for an infinite population:

$$\begin{cases} \frac{ds_\alpha}{dt} = \Pi R^{-1} s_\alpha + R_\alpha, \\ \frac{d\bar{z}_\alpha}{dt} = -R^{-1}\Pi \bar{z}_\alpha - R^{-1} s_\alpha, \qquad \alpha \in [\underline{\alpha}, \overline{\alpha}] \\ \bar{r}_\alpha(t) = \int_{\alpha' \in [\underline{\alpha}, \overline{\alpha}]} \bar{z}_{\alpha'}(t) dF_\alpha(\alpha'), \\ R_\alpha = \gamma \bar{r}_\alpha, \end{cases} \tag{21}$$

where $\Pi > 0$ is the solution of ARE:

$$-\Pi R^{-1} \Pi + Q = 0. \tag{22}$$

(C5) Assume $\gamma < 1$.

Theorem Under **(C1)**-**(C5)**, there exists a unique bounded solution $(s_\alpha(\cdot), \bar{z}_\alpha(\cdot), r_\alpha(\cdot))$ to the NCE equation system 21.

Part IX – Stochastic Consensus Problem by NCE (cnt)

Finite population pre-computable NCE consensus equations $(1 \le i \le N)$, from (21)

$$\begin{cases} \frac{ds_i}{dt} = \Pi R^{-1} s_i + \Phi_i, \\ \frac{d\bar{z}_i}{dt} = -R^{-1}\Pi \bar{z}_i - R^{-1} s_i, \\ \Phi_i = \gamma \sum_{j=1}^{N} \omega_{p_i p_j}^{(N)} \bar{z}_j. \end{cases} \quad (23)$$

When s_i is in the steady state:

$$\frac{d\bar{z}_i}{dt} = -R^{-1}\Pi \bar{z}_i + \gamma \Pi^{-1} \sum_{j=1}^{N} \omega_{p_i p_j}^{(N)} \bar{z}_j. \quad (24)$$

Set $Q = I$ then from (22) $R^{-1}\Pi = \Pi^{-1}$, and

$$\frac{d\bar{z}_i}{dt} = R^{-1}\Pi(-\bar{z}_i + \gamma \sum_{j=1}^{N} \omega_{p_i p_j}^{(N)} \bar{z}_j). \quad (25)$$

Part IX – Stochastic Consensus Problem by NCE (cnt)

NCE consensus equation dynamics [pre-computed feedback];

$$\frac{d\bar{z}}{dt} = -R^{-1}\Pi G\bar{z}, \tag{26}$$

where $z \in \mathbb{R}^{Nn}$ and

$$(G)_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\gamma\omega_{p_i p_j}^{(N)} & \text{otherwise.} \end{cases}$$

For $\gamma = 1$, G is a normalized Laplacian matrix.

NCE finite population stochastic consensus dynamics with pre-computed feedback (26):

$$dz_i = R^{-1}\Pi(-z_i + \gamma\sum_{j=1}^{N}\omega_{p_i p_j}^{(N)}\bar{z}_j)dt + CdW_i.$$

Part IX – Stochastic Consensus Problem by NCE (cnt)

Assume $Q = R = I$ and $\gamma = 1$, then there exists a unique, bounded solution to $(1 \leq i \leq N)$

$$\begin{cases} \frac{d\bar{z}_i}{dt} = -(\bar{z}_i - \Phi_i), \\ \Phi_i = \sum_{j=1}^{N} \omega_{p_i p_j}^{(N)} \bar{z}_j, \end{cases} \tag{27}$$

or equivalently

$$\frac{d\bar{z}}{dt} = -L\bar{z}, \ \ \bar{z}(0) \text{ given.}$$

In general

$$\lim_{t\to\infty} (\bar{z}_i(t) - \bar{z}_j(t)) = 0, \ \forall 1 \leq i, j \leq N.$$

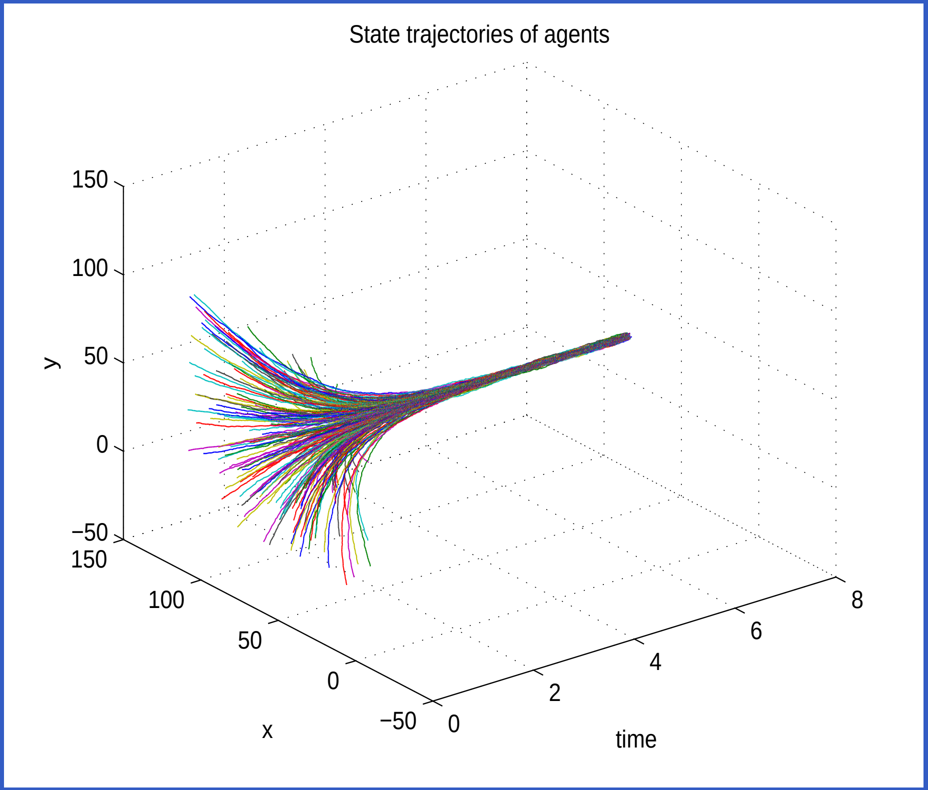
and for a doubly stochastic Ω

$$\lim_{t\to\infty} \bar{z}_i(t) = \frac{1}{N} \sum_{j=1}^{N} \bar{z}_j(0), \ \forall 1 \leq i \leq N.$$

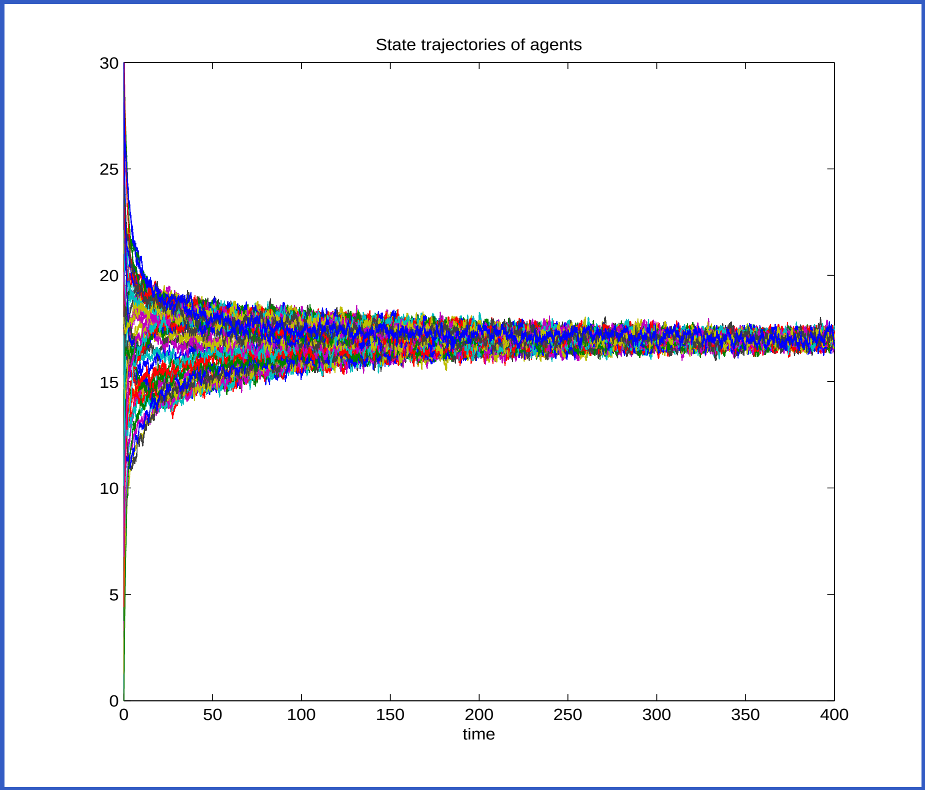
Part IX – Stochastic Consensus Problem by NCE (cnt)

- Infinite Population Stochastic NCE Consensus Problem:
  - Prior information $(F_\alpha(\cdot), \bar{z}_\alpha(0), \alpha \in [\underline{\alpha}, \overline{\alpha}])$ available to all agents.
  - Deterministic pre-computable "global feedback".
  - Nash interpretation.
- Finite Population Stochastic NCE Consensus Problem:
  - Prior information $(\Omega^N, \bar{z}_\alpha^N(0), \alpha \in [\underline{\alpha}, \overline{\alpha}])$ available to all agents.
  - Deterministic pre-computable "local feedback".
  - Nash interpretation.
- Finite Population Deterministic Consensus Algorithms:
  - No prior information.
  - Local communications (Laplacian feedback).
  - No Nash interpretation.

Part IX – Simulations



(a) Clique graph.

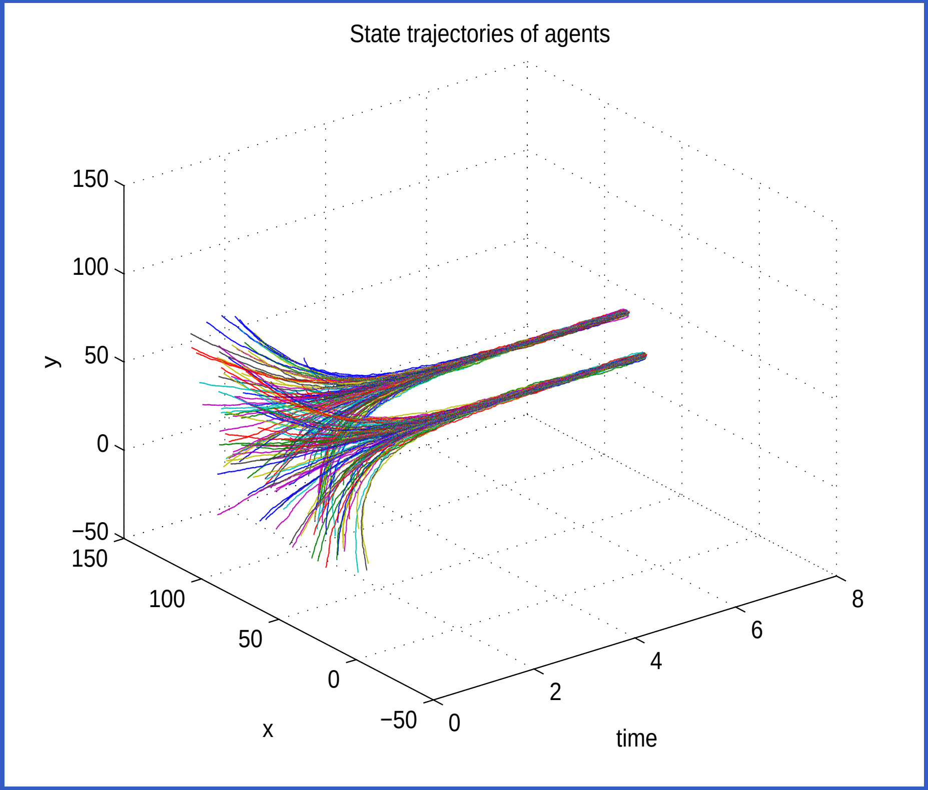


(b) Circular graph.

- Convergence in (a) is faster than (b).

Part IX – Simulations (cnt)

- For disconnected graphs we have the convergence of each group.



(c) Disconnected graph with two connected groups.

Part IX – NCE Consensus and Standard Consensus

- Stochastic with a cost function verses deterministic without the cost.
- We derive a consensus seeking Laplacian feedback from the NCE equations.
- Obtain convergence of the $\bar{z}(t)$ for all $\Omega = (\omega^{(N)}_{p_i p_j})$ which satisfy the localization conditions.
- Obtain convergence of each subgroup for disconnected graphs.
- There exists a duality between the a priori information needed by the NCE approach for constructing the pre-computed decentralized control laws and the local information exchanges between agents in the standard consensus algorithm.
- In the NCE-Consensus formulation, each agent's behaviour is optimal with respect to other agents in a game theoretic Nash sense.

Concluding Remarks

- A theory for decentralized decision-making with many competing agents
- Control synthesis via NCE methodology. Consequences for Rational Expectations and Macroeconomic Policy?
- Existence of asymptotic equilibria (first in population then in time)
- Application to network call admission control (e.g. Ma, Malhamé, PEC)
- Ideas closely related to the physics of interacting particle systems.
- Suggest a convergence of control theory, multi-agent systems theory and statistical physics into a

  cybernetic-math physics synthesis

  for mass competitive-cooperative decision problems.